ON THE CURVATURE OF CONFORMAL TRANSFORMATION OF RIEMANNIAN MANIFOLD

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ABSTRACT. In this article, we mainly discussed the relations of curvature tensors of a Riemannian metric under a conformal transformation.

1. MAIN RESULTS

Suppose (M, g) is a n-dimensional Riemannian manifold. Consider a conformal transformation of a Riemannian structure

$$\bar{g}_{ij} = e^{2\rho} g_{ij}.$$

Then the connection coefficients $\bar{\Gamma}^l_{ij}$ corresponding to $\bar{g}_{ij}$ will be

$$\begin{aligned} \bar{\Gamma}^l_{ij} &= \Gamma^l_{ij} + \rho_j \delta^l_i + \rho_i \delta^l_j - \rho_k g^{lk} g_{ij} \\ &= \Gamma^l_{ij} + \rho_j \delta^l_i + \rho_i \delta^l_j - \rho^l g_{ij}, \end{aligned} \tag{1}$$

where

$$\rho_i = \frac{\partial \rho}{\partial u^i}, \quad \rho^l = g^{lk} \rho_k.$$

From (1), we have

$$\bar{\mathsf{Rm}}_j{}^i{}_{kl} = \mathsf{Rm}_j{}^i{}_{kl} - \rho_{jk} \delta^i_l + \rho_{jl} \delta^i_k - g_{jk} \rho^i_l + g_{jl} \rho^i_k, \tag{2}$$

where

$$\rho_{jk} = \rho_{j,k} - \rho_j \rho_k + \frac{1}{2} g^{\alpha\beta} \rho_\alpha \rho_\beta g_{jk}.$$

Hence

$$\bar{\mathsf{Rc}}_{jk} = \mathsf{Rc}_{jk} - (n-2)\rho_{jk} - g_{jk} \rho^\alpha_\alpha, \tag{3}$$

and

$$\bar{\mathsf{Rs}} = e^{-2\rho}(\mathsf{Rs} - 2(n-1)\rho^\alpha_\alpha). \tag{4}$$

Here $\bar{\mathsf{Rm}}_j{}^i{}_{kl}$, $\bar{\mathsf{Rc}}_{jk}$ and $\bar{\mathsf{Rs}}$ denote the curvature tensor, the Ricci tensor and the scalar curvature, respectively, of the new structure. This is the preliminary of H. Yamabe's thesis[Yam60].

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2. The Proofs

We know there exist a unique torsion-free and metric compatible connection, ie. the Riemannian connection, on a Riemannian manifold, and the connection coefficients Γ_{ikj} can be expressed exactly by the metric coefficients g_{ij} as

$$\Gamma_{ikj} = \frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right\}.$$

Since $\bar{g}_{ij} = e^{2\rho} g_{ij}$, thus

$$\bar{g}^{ik} = e^{-2\rho} g^{ik}, \quad \frac{\partial \bar{g}_{ik}}{\partial u^j} = e^{2\rho}\left(2\rho_j g_{ik} + \frac{\partial g_{ik}}{\partial u^j}\right),$$

and

$$\begin{aligned}\bar{\Gamma}_{ikj} &= \frac{1}{2}\left\{\frac{\partial \bar{g}_{ik}}{\partial u^j} + \frac{\partial \bar{g}_{kj}}{\partial u^i} - \frac{\partial \bar{g}_{ij}}{\partial u^k}\right\} \\ &= \frac{1}{2}e^{2\rho}\left\{2\rho_j g_{ik} + 2\rho_i g_{kj} - 2\rho_k g_{ij} + \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right\} \\ &= e^{2\rho}\left\{\Gamma_{ikj} + \rho_j g_{ik} + \rho_i g_{kj} - \rho_k g_{ij}\right\},\end{aligned}$$

As $\bar{\Gamma}^l_{ij} = \bar{g}^{lk}\bar{\Gamma}_{ikj}$, then

$$\begin{aligned}\bar{\Gamma}^l_{ij} &= g^{lk}\left\{\Gamma_{ikj} + \rho_j g_{ik} + \rho_i g_{kj} - \rho_k g_{ij}\right\} \\ &= \Gamma^l_{ij} + \rho_j \delta^l_i + \rho_i \delta^l_j - \rho_k g^{lk} g_{ij},\end{aligned}$$

thus, we have obtained (1).

Define $R(X,Y)\colon \Gamma(TM) \to \Gamma(TM)$ as

$$\begin{aligned}R(X,Y)Z &\stackrel{\text{def}}{=} \left\{[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right\} Z \\ &\stackrel{\text{def}}{=} \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\end{aligned}$$

where $X, Y, Z \in \Gamma(TM)$, and set $\mathsf{Rm}_i{}^l{}_{jk}$ be the component of R, ie.,

$$R\left(\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right)\frac{\partial}{\partial u^i} = \mathsf{Rm}_i{}^l{}_{jk}\frac{\partial}{\partial u^l},$$

and define

$$\mathrm{R}(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle,$$

then the component of R will be

$$\mathsf{Rm}_{hijk} = g_{hl}\mathsf{Rm}_i{}^l{}_{jk} = \langle R\left(\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right)\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^h}\rangle.$$

the Einstein summation rule has been used. Note

$$\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j} = \Gamma^k_{ji}\frac{\partial}{\partial u^k}, \quad \left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right] = 0,$$

we have

$$\mathsf{Rm}_i{}^l{}_{jk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^h_{ik}\Gamma^l_{hj} - \Gamma^h_{ij}\Gamma^l_{hk}.$$

In fact, just by the definition of $\mathsf{Rm}_i{}^l{}_{jk}$ we have

$$\begin{aligned}
\mathsf{Rm}_i{}^l{}_{jk}\frac{\partial}{\partial u^l} &= R(\frac{\partial}{\partial u^j},\frac{\partial}{\partial u^k})\frac{\partial}{\partial u^i} \\
&= \nabla_{\frac{\partial}{\partial u^j}}\nabla_{\frac{\partial}{\partial u^k}}\frac{\partial}{\partial u^i} - \nabla_{\frac{\partial}{\partial u^k}}\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i} - \nabla_{[\frac{\partial}{\partial u^k},\frac{\partial}{\partial u^j}]}\frac{\partial}{\partial u^i} \\
&= \nabla_{\frac{\partial}{\partial u^j}}\nabla_{\frac{\partial}{\partial u^k}}\frac{\partial}{\partial u^i} - \nabla_{\frac{\partial}{\partial u^k}}\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i} \\
&= \nabla_{\frac{\partial}{\partial u^j}}\left(\Gamma^l_{ik}\frac{\partial}{\partial u^l}\right) - \nabla_{\frac{\partial}{\partial u^k}}\left(\Gamma^l_{ij}\frac{\partial}{\partial u^l}\right) \\
&= \frac{\partial\Gamma^l_{ik}}{\partial u^j}\frac{\partial}{\partial u^l} + \Gamma^h_{ik}\Gamma^l_{hj}\frac{\partial}{\partial u^l} - \left(\frac{\partial\Gamma^l_{ij}}{\partial u^k}\frac{\partial}{\partial u^l} + \Gamma^h_{ij}\Gamma^l_{hk}\frac{\partial}{\partial u^l}\right) \\
&= \left\{\frac{\partial\Gamma^l_{ik}}{\partial u^j} - \frac{\partial\Gamma^l_{ij}}{\partial u^k} + \Gamma^h_{ik}\Gamma^l_{hj} - \Gamma^h_{ij}\Gamma^l_{hk}\right\}\frac{\partial}{\partial u^l},
\end{aligned}$$

form which we can deduce the formula of $\mathsf{Rm}_i{}^l{}_{jk}$.

In order to get the formula of Rm_{hijk}, we shall note the following relation

$$g_{hl}\frac{\partial\Gamma^l_{ik}}{\partial u^j} = \frac{\partial\Gamma_{ihk}}{\partial u^j} - \Gamma^l_{ik}\frac{\partial g_{hl}}{\partial u^j} = \frac{\partial\Gamma_{ihk}}{\partial u^j} - \Gamma^l_{ik}\left(\Gamma_{hlj} + \Gamma_{lhj}\right).$$

then,

$$\begin{aligned}
\mathsf{Rm}_{hijk} = g_{hl}\mathsf{Rm}_i{}^l{}_{jk} &= g_{hl}\left\{\frac{\partial\Gamma^l_{ik}}{\partial u^j} - \frac{\partial\Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}\right\} \\
&= \frac{\partial\Gamma_{ihk}}{\partial u^j} - \Gamma^l_{ik}(\Gamma_{hlj} + \Gamma_{lhj}) - \left(\frac{\partial\Gamma_{ihj}}{\partial u^k} - \Gamma^l_{ij}(\Gamma_{hlk} + \Gamma_{lhk})\right) \\
&\quad + \Gamma^p_{ik}\Gamma_{phj} - \Gamma^p_{ij}\Gamma_{phk} \\
&= \frac{\partial\Gamma_{ihk}}{\partial u^j} - \frac{\partial\Gamma_{ihj}}{\partial u^k} + \Gamma^l_{ij}\Gamma_{hlk} - \Gamma^l_{ik}\Gamma_{hlj}.
\end{aligned}$$

Moreover, as Γ_{ihk} can be expressed by g_{ij}, we have

$$\begin{aligned}
\mathsf{Rm}_{hijk} &= \frac{\partial\Gamma_{ihk}}{\partial u^j} - \frac{\partial\Gamma_{ihj}}{\partial u^k} + \Gamma^l_{ij}\Gamma_{hlk} - \Gamma^l_{ik}\Gamma_{hlj} \\
&= \frac{1}{2}\frac{\partial}{\partial u^j}\left\{\frac{\partial g_{ih}}{\partial u^k} + \frac{\partial g_{hk}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^h}\right\} - \frac{1}{2}\frac{\partial}{\partial u^k}\left\{\frac{\partial g_{ih}}{\partial u^j} + \frac{\partial g_{hj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^h}\right\} \\
&\quad + \Gamma^l_{ij}\Gamma_{hlk} - \Gamma^l_{ik}\Gamma_{hlj} \\
&= \frac{1}{2}\left\{\frac{\partial^2 g_{hk}}{\partial u^i\partial u^j} + \frac{\partial^2 g_{ij}}{\partial u^h\partial u^k} - \frac{\partial^2 g_{ik}}{\partial u^h\partial u^j} - \frac{\partial^2 g_{hj}}{\partial u^i\partial u^k}\right\} + \Gamma^l_{ij}\Gamma_{hlk} - \Gamma^l_{ik}\Gamma_{hlj}
\end{aligned}$$

All the above formulas are needed to calculate the Riemannian curvature tensor. Let's turn to the calculation of $\bar{\mathsf{Rm}}_i{}^l{}_{jk}$, I have done it at the assistance of Mathematica.

First, define the following functions in Mathematica:

```
bgamma2[k_][i_, j_] :=gamma2[k][i, j]
       + f[i] \[Delta][k, j] + f[j] \delta[k, i]
         -fu[k] g[i, j];
Dbgamma2[k_][i_, j_][l_] :=Dgamma2[k][i, j][l]
       + f[i, l] \[Delta][k, j] + f[j, l] \[Delta][k, i]
       - fu[k][l] g[i, j] - fu[k] Dg[i, j][l];
bRm2[l_][i_, j_, k_][s_] :=Dbgamma2[l][i, k][j]
  - Dbgamma2[l][i, j][k]
   + bgamma2[s][i, k] bgamma2[l][s, j]
      - bgamma2[s][i, j] bgamma2[l][s, k]
```

Then type `bRm2[l][i, j, k][s] // ExpandAll` in Mathematica, after a simple calculation we get[†]

$$\begin{aligned}\bar{\mathsf{Rm}}_i{}^l{}_{jk} = \mathsf{Rm}_i{}^l{}_{jk} &- \rho_s\rho^s g_{ik}\delta^l_j + \rho_k\rho_i\delta^l_j - \rho_{ik}\delta^l_j + \rho_s\Gamma^s_{ik}\delta^l_j \\ &+ \rho_s\rho^s g_{ij}\delta^l_k - \rho_i\rho_j\delta^l_k + \rho_{ij}\delta^l_k - \rho_s\Gamma^s_{ij}\delta^l_k \\ &+ \rho_j\rho^l g_{ik} - \rho^s g_{ik}\Gamma^l_{sj} - g_{ik}\rho^l_j \\ &- \rho_k\rho^l g_{ij} + \rho^s g_{ij}\Gamma^l_{sk} + g_{ij}\rho^l_k \\ &+ \rho^l\frac{\partial g_{ij}}{\partial u^k} + \rho^l g_{sk}\Gamma^s_{ij} - \rho^l\frac{\partial g_{ik}}{\partial u^j} - \rho^l g_{sj}\Gamma^s_{ik}.\end{aligned}$$

The last four terms will be vannish, and note that, for ρ as a prue function, the corvariant derivative $\rho_{;i}$ and $\rho_{;ij}$ will be

$$\rho_{;i} = \frac{\partial\rho}{\partial u^i}, \quad \rho_{;ik} = \frac{\partial^2\rho}{\partial u^i\partial u^k} - \rho_k\Gamma^k_{ik} \overset{\text{def}^\ddagger}{=} \rho_{ik} - \rho_k\Gamma^k_{ik},$$

then

$$\rho_s\Gamma^s_{ik}\delta^l_j = (\rho_{ik} - \rho_{;ik})\delta^l_j,$$

ie.,

$$\begin{aligned}-\rho_{;ik}\delta^l_j &= -\rho_{ik}\delta^l_j + \rho_s\Gamma^s_{ik}\delta^l_j, \\ \rho_{;ij}\delta^l_k &= \rho_{ij}\delta^l_k - \rho_s\Gamma^s_{ij}\delta^l_k,\end{aligned}$$

thus,

$$\begin{aligned}\bar{\mathsf{Rm}}_i{}^l{}_{jk} = \mathsf{Rm}_i{}^l{}_{jk} &- \rho_s\rho^s g_{ik}\delta^l_j + \rho_k\rho_i\delta^l_j - \rho_{;ik}\delta^l_j \\ &+ \rho_s\rho^s g_{ij}\delta^l_k - \rho_i\rho_j\delta^l_k + \rho_{;ij}\delta^l_k \\ &- g_{ik}\left(\rho^s\Gamma^l_{sj} + \rho^l_j - \rho_j\rho^l\right) \\ &+ g_{ij}\left(\rho^s\Gamma^l_{sk} + \rho^l_k - \rho_k\rho^l\right).\end{aligned}$$

[†]The output of Mathematica can be handled by a Macro of WinEdt to get the format of TeX.

[‡]It should be noted that there the definition of ρ_{ik} is different from the definition in [Yam60], see section 1.

Note

$$\rho_k^l \overset{\mathrm{def}^\dagger}{=} \frac{\partial \left(g^{lh}\rho_h\right)}{\partial u^k} = \frac{\partial g^{lh}}{\partial u^k}\rho_h + g^{lh}\rho_{hk},$$

$$\begin{aligned}\frac{\partial g^{hl}}{\partial u^k} &= g^{hs}g_{sp}\frac{\partial g^{pl}}{\partial u^k} = -g^{hs}g^{pl}\frac{\partial g_{sp}}{\partial u^k}\\ &= -g^{hs}g^{pl}\left(\Gamma_{spk} + \Gamma_{psk}\right)\\ &= -\left(g^{hs}\Gamma_{sk}^l + g^{pl}\Gamma_{pk}^h\right),\end{aligned}$$

and

$$\rho_{;hk} = \rho_{hk} - \rho_s\Gamma_{hk}^s,$$

we have

$$\begin{aligned}\rho^s\Gamma_{sk}^l &= -g^{lh}\rho_s\Gamma_{hk}^s + \rho^s\Gamma_{sk}^l + \rho_h g^{pl}\Gamma_{pk}^h\\ &= -g^{lh}\rho_s\Gamma_{hk}^s - \rho_h\frac{\partial g_{hl}}{\partial u^k}\\ &= g^{lh}(\rho_{;hk} - \rho_{hk}) - \rho_h\frac{\partial g_{hl}}{\partial u^k}\\ &= g^{lh}\rho_{;hk} - \rho_k^l,\end{aligned}$$

thus,

$$g_{ij}\left(\rho^s\Gamma_{sk}^l + \rho_k^l\right) = g_{ij}g^{lh}\rho_{;hk},$$

$$g_{ij}\left(\rho^s\Gamma_{sk}^l + \rho_k^l - \rho_k\rho^l\right) = g_{ij}g^{lh}(\rho_{;hk} - \rho_k\rho_h),$$

similarly,

$$g_{ik}\left(\rho^s\Gamma_{sj}^l + \rho_j^l - \rho_j\rho^l\right) = g_{ik}g^{lh}(\rho_{;hj} - \rho_j\rho_h).$$

Finally, if we set $\rho_{,ij} = \rho_{;ij} - \rho_i\rho_j + \rho_s\rho^s g_{ij}$, then

(5)
$$\bar{\mathsf{Rm}}_i{}^l{}_{jk} = \mathsf{Rm}_i{}^l{}_{jk} + \delta_k^l\rho_{,ij} - \delta_j^l\rho_{,ik} + g_{ij}g^{lh}(\rho_{;hk} - \rho_k\rho_h) - g_{ik}g^{lh}(\rho_{;hj} - \rho_j\rho_h).$$

If we reset $\rho_{hk} \overset{\mathrm{def}}{=} \rho_{;hk} - \rho_h\rho_k + \frac{1}{2}\rho_s\rho^s g_{hk}$, and $\rho_k^l \overset{\mathrm{def}}{=} g^{lh}\rho_{hk}$ as in the article [Yam60], then it's easy to check the following formula

(6)
$$\bar{\mathsf{Rm}}_i{}^l{}_{jk} = \mathsf{Rm}_i{}^l{}_{jk} + \rho_{ij}\delta_k^l - \rho_{ik}\delta_j^l + g_{ij}\rho_k^l - g_{ik}\rho_j^l.$$

Equation (6) and the corresponding formula in article[Yam60] is slightly different, mainly because we define the $\mathsf{Rm}_i{}^l{}_{jk}$ and Rm_{ijkl} different, this definition assured that the unit sphere with curvature 1, see [Eis97].

Next, we will deduce the Ricci curvature and scalar curvature.

The Ricci curvature in the direction $X \in \Gamma(TM)$ is defined as

$$\mathsf{Ric}(X,X) = g^{jl}\langle R(X,\frac{\partial}{\partial u^j})\frac{\partial}{\partial u^l}, X\rangle.$$

[†] It should be noted that there the definition of ρ_k^l is different from the definition in [Yam60], in which ρ_k^l is defined as $g^{ls}\rho_{sk}$, see section 1.

The Ricci tensor is

$$\mathsf{Rc}_{ik} = g^{hj}\mathsf{Rm}_{hijk}.$$

By (6),

$$\begin{aligned}\bar{\mathsf{Rc}}_{ik} &= \bar{g}^{hj}\bar{\mathsf{Rm}}_{hijk} = g^{hj}g_{hl}\left(\mathsf{Rm}_i{}^l{}_{jk} + \rho_{ij}\delta^l_k - \rho_{ik}\delta^l_j + g_{ij}\rho^l_k - g_{ik}\rho^l_j\right)\\ &= \mathsf{Rc}_{ik} + \rho_{ik} - n\rho_{ik} + \rho_{ik} - g_{ik}\rho^l_l\\ &= \mathsf{Rc}_{ik} - (n-2)\rho_{ik} - g_{ik}\rho^l_l.\end{aligned}$$

The scalar curvature is defined as

$$\mathsf{Rs} = g^{ik}\mathsf{Rc}_{ik}.$$

so,

$$\begin{aligned}\bar{\mathsf{Rs}} &= \bar{g}^{ik}\bar{\mathsf{Rc}}_{ik} = \bar{g}^{ik}\left(\mathsf{Rc}_{ik} - (n-2)\rho_{ik} - g_{ik}\rho^l_l\right)\\ &= e^{-2\rho}\left(\mathsf{Rs} - (n-2)\rho^k_k - n\rho^l_l\right)\\ &= e^{-2\rho}\left(\mathsf{Rs} - 2(n-1)\rho^l_l\right).\end{aligned}$$

Finally, it easy to verify that the Weyl's conformal curvature tensor

$$\begin{aligned}\mathsf{C}_i{}^l{}_{jk} = \mathsf{Rm}_i{}^l{}_{jk} &+ \frac{1}{n-2}\left(-\delta^l_j\mathsf{Rc}_{ik} + \delta^l_k\mathsf{Rc}_{ij} - g_{ik}\mathsf{Rc}^l_j + g_{ij}\mathsf{Rc}^l_k\right)\\ &+ \frac{\mathsf{Rs}}{(n-1)(n-2)}\left(\delta^l_k g_{ij} - \delta^l_j g_{ik}\right), \quad \mathsf{Rc}^l_j = g^{ls}\mathsf{Rc}_{sj},\end{aligned}$$

is a conformal invariant.

For further results on conformal transformation we refer to [Eis97, Kul70a, Kul70b].

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