Valuations on Sobolev Spaces

Monika Ludwig *

Abstract

All affinely covariant convex-body-valued valuations on the Sobolev space $W^{1,1}(\mathbb{R}^n)$ are completely classified. It is shown that there is a unique such valuation for Blaschke addition. This valuation turns out to be the operator which associates with each function $f \in W^{1,1}(\mathbb{R}^n)$ the unit ball of its optimal Sobolev norm.

2000 AMS subject classification: 46B20 (46E35, 52A21,52B45)

Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$ that is normalized so that its unit ball has the same volume, v_n, as the n-dimensional Euclidean unit ball. For such a norm, the sharp Gagliardo-Nirenberg-Sobolev inequality states that

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|_* \, dx \geqslant n \, v_n^{1/n} \, |f|_{\frac{n}{n-1}} \tag{1}$$

for every $f \in W^{1,1}(\mathbb{R}^n)$. Here for $p \geqslant 1$, $|f|_p$ denotes the L^p norm of f and $\|\cdot\|_*$ the dual norm of $\|\cdot\|$ (see Section 1 for precise definitions). The Sobolev space $W^{1,1}(\mathbb{R}^n)$ is the space of functions $f \in L^1(\mathbb{R}^n)$ such that their weak gradient ∇f is in $L^1(\mathbb{R}^n)$. If the unit ball B of $\|\cdot\|$ is the Euclidean unit ball, then inequality (1) goes back to Federer and Fleming [15] and Maz′ya [46] and is known to be equivalent to the Euclidean isoperimetric inequality. For general norms, (1) was established by Gromov [49, Appendix]. Note that the right hand side of (1) does not depend on $\|\cdot\|$. Hence for a given $f \in W^{1,1}(\mathbb{R}^n)$, $n \geqslant 2$, we may ask for its *optimal Sobolev norm*, that is, for the norm that minimizes the left-hand side of (1) among all norms whose unit balls have volume v_n.

This natural and important question was first asked by Lutwak, Yang and Zhang in [45]. They showed that the unit ball $\langle f \rangle$ corresponding to the optimal Sobolev norm of $f \in W^{1,1}(\mathbb{R}^n)$ is (up to normalization) the unique origin-symmetric convex body (that is, compact, convex set) in $\mathbb{R}^n$ such that

$$\int_{S^{n-1}} g(u) \, dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(-\nabla f(x)) \, dx \tag{2}$$

for every even $g \in C(\mathbb{R}^n)$ that is positively homogeneous of degree 1. Here $S(K, \cdot)$ is the Aleksandrov-Fenchel-Jessen surface area measure of $K \in \mathcal{K}_c^n$ and $\mathcal{K}_c^n$ is the set of origin-symmetric convex bodies in $\mathbb{R}^n$ with non-empty interiors together with the convex body $\{0\}$. The equations (2) are a functional version of the classical even Minkowski problem and define

*Research supported, in part, by NSF grant DMS-0805623

an operator $\langle\cdot\rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ which associates with each $f \in W^{1,1}(\mathbb{R}^n)$ its *optimal Sobolev body* $\langle f\rangle$. Thus (2) provides a second description of the optimal Sobolev norm. Lutwak, Yang and Zhang [45] showed that the optimal Sobolev body corresponds also to the optimal norm in a family of sharp Gagliardo-Nirenberg inequalities recently established by Cordero, Nazaret, and Villani [14]. Moreover, the optimal Sobolev body has proved to be critical in recent results on affine isoperimetric inequalities (see [13, 24, 40, 44, 45, 59, 60]).

Using valuations on Sobolev spaces, we obtain a new and totally different description of the operator $f \mapsto \langle f\rangle$. A function z defined on a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an abelian semigroup is called a *valuation* if

$$\mathrm{z}(f \vee g) + \mathrm{z}(f \wedge g) = \mathrm{z}(f) + \mathrm{z}(g) \tag{3}$$

for all $f, g \in \mathcal{L}$. A function z defined on some subset $\mathcal{M}$ of $\mathcal{L}$ is called a valuation on $\mathcal{M}$ if (3) holds whenever $f, g, f \vee g, f \wedge g \in \mathcal{M}$.

Investigations of valuations on convex bodies $(\mathcal{K}^n, \cup, \cap)$ have been an active and prominent part of mathematics ever since Dehn's solution of Hilbert's Third Problem in 1900. Blaschke obtained the first classification of real-valued valuations on convex bodies that are $\mathrm{SL}(n)$ invariant in the 1930s. This was greatly extended by Hadwiger in his famous classification of continuous, rigid motion invariant valuations and characterization of elementary mixed volumes. See [25, 30, 47, 48] for information on the classical theory of valuations on convex bodies and [1–5, 9, 16, 20–23, 33–35, 38, 39, 51, 53, 54, 58] for some of the more recent results. Valuations were also investigated on star shaped sets [27, 28], on manifolds [6–8, 10, 11] and on Lebesgue spaces [37, 56, 57].

In this paper, we classify valuations on $(W^{1,1}(\mathbb{R}^n), \vee, \wedge)$, where for $f, g \in W^{1,1}(\mathbb{R}^n)$, the function $f \vee g$ denotes the pointwise maximum and the function $f \wedge g$ the pointwise minimum of f and g. As in the classical results for valuations on convex bodies we use invariance and covariance properties to obtain characterizations of important operators. An operator $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is called $\mathrm{GL}(n)$ *covariant* if for some $p \in \mathbb{R}$,

$$\mathrm{z}(f \circ \phi^{-1}) = |\det \phi|^p\, \phi\, \mathrm{z}(f)$$

for all $f \in W^{1,1}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$, where $\det \phi$ is the determinant of ϕ. An operator z is called *translation invariant* if $\mathrm{z}(f \circ \tau^{-1}) = \mathrm{z}(f)$ for all $f \in W^{1,1}(\mathbb{R}^n)$ and translations τ. It is called *homogeneous* if for some $q \in \mathbb{R}$, we have $\mathrm{z}(sf) = |s|^q\, \mathrm{z}(f)$ for all $f \in W^{1,1}(\mathbb{R}^n)$ and $s \in \mathbb{R}$. An operator $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is called *affinely covariant* if z is homogeneous, translation invariant and $\mathrm{GL}(n)$ covariant.

Theorem 1. *An operator* $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, \#\rangle$*, where* $n \geqslant 3$*, is a continuous, affinely covariant valuation if and only if there is a constant* $c \geqslant 0$ *such that*

$$\mathrm{z}(f) = c\langle f\rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$.

Here $\#$ denotes Blaschke addition on $\mathcal{K}_c^n$, that is, for $K, L \in \mathcal{K}_c^n$, the convex body $K \# L$ is the (uniquely determined) origin-symmetric convex body such that $S(K \# L, \cdot) = S(K, \cdot) + S(L, \cdot)$ (see Section 1 for precise definitions). See [12, 18, 26, 29, 41–43, 52] for some of the recent results involving Blaschke addition and, in particular, Haberl [21], where a classification of Blaschke valuations on convex bodies was obtained.

Theorem 1 is in a certain sense dual to the following classification result for valuations $\mathrm{z}: W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$. Here $+$ denotes Minkowski addition on $\mathcal{K}_c^n$, that is, for $K, L \in \mathcal{K}_c^n$, we have $K + L = \{x + y : x \in K, y \in L\}$. We say that an operator $\mathrm{z}: W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is $\mathrm{GL}(n)$ *contravariant* if for some $p \in \mathbb{R}$,

$$\mathrm{z}(f \circ \phi^{-1}) = |\det \phi|^p \, \phi^{-t} \, \mathrm{z}(f)$$

for all $f \in W^{1,1}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$, where ϕ^{-t} is the transpose of the inverse of ϕ. An operator $\mathrm{z}: W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is called *affinely contravariant* if z is homogeneous, translation invariant and $\mathrm{GL}(n)$ contravariant.

Theorem 2. *An operator* $\mathrm{z}: W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$*, where* $n \geqslant 3$*, is a continuous, affinely contravariant valuation if and only if there is a constant* $c \geqslant 0$ *such that*

$$\mathrm{z}(f) = c \, \Pi \, \langle f \rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$*.*

Here ΠK denotes the *projection body* of a convex body K. Projection bodies were introduced by Minkowski at the turn of the last century and have proved to be very useful in many ways and subjects (cf. [17]). They can be defined in the following way. Every convex body K is uniquely determined by its support function $h(K, \cdot)$, where $h(K, v) = \max\{v \cdot x : x \in K\}$ for $v \in \mathbb{R}^n$ and $v \cdot x$ is the standard inner product of $v, x \in \mathbb{R}^n$. The projection body of K is the convex body whose support function is given by

$$h(\Pi K, v) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS(K, u), \quad v \in \mathbb{R}^n.$$

Combined with (2), this gives

$$h(\Pi \langle f \rangle, v) = \frac{1}{2} \int_{\mathbb{R}^n} |v \cdot \nabla f(x)| \, dx. \tag{4}$$

Also the convex body $\Pi \langle f \rangle$ has proved to be critical for affine isoperimetric inequalities. In particular, the affine Zhang-Sobolev inequality [60] is a volume inequality for the polar body of $\Pi \langle f \rangle$ which strengthens and implies the Euclidean case of the Sobolev inequality (1).

1 Background material on convex bodies

General references on convex bodies are the books by Gardner [17], Gruber [19], Schneider [50], and Thompson [55]. We work in Euclidean n-space, $\mathbb{R}^n$, and write $x = (x_1, \dots, x_n)$ for $x \in \mathbb{R}^n$. Throughout this paper, $u \cdot x$ denotes the standard inner product of $u, x \in \mathbb{R}^n$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. The vectors of the standard basis of $\mathbb{R}^n$ are denoted by $e_1, \dots, e_n$ and the k-dimensional volume of a k-dimensional convex body F by $V_k(F)$.

Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$. The subspace of convex bodies with non-empty interiors which contain the origin is denoted by $\mathcal{K}_0^n$ and the subspace of origin-symmetric bodies with non-empty interiors by $\mathcal{K}_c^n$. These spaces are equipped with the Hausdorff metric δ defined by

$$\delta(K, L) = \max\{|h(K, u) - h(L, u)| : u \in S^{n-1}\}.$$

Minkowski addition can also be described by support functions, since

$$h(K + L, v) = h(K, v) + h(L, v) \tag{5}$$

for all $K, L \in \mathcal{K}^n$ and $v \in \mathbb{R}^n$. Note that $\langle \mathcal{K}_c^n, + \rangle$ is an abelian semigroup.

Blaschke addition is defined using the surface area measure $S(K, \cdot)$ for $K \in \mathcal{K}_0^n$. For a Borel set $\omega \subset S^{n-1}$, the surface area measure $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a unit normal vector of K belonging to ω. The solution to the Minkowski problem (see [50]) states that a non-negative Borel measure μ on S^{n-1} is the surface area measure of a convex body if and only if μ is not concentrated on a great subsphere and has its centroid, $\frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} u \, d\mu(u)$, at the origin. If such a measure μ is given, there is a unique convex body $K \in \mathcal{K}_0^n$ with surface area measure $S(K, \cdot) = \mu$ that has its centroid, $\frac{1}{V_n(K)} \int_K x \, dx$, at the origin. For $K, L \in \mathcal{K}_0^n$, their Blaschke sum, $K \# L$, is defined as the unique convex body with centroid at the origin such that

$$S(K \# L, \cdot) = S(K, \cdot) + S(L, \cdot).$$

Since the sum of two surface area measures satisfies the necessary conditions of the Minkowski problem, Blaschke addition is well defined by the solution of the Minkowski problem. For $t > 0$ and $K \in \mathcal{K}_0^n$, the Blaschke multiple, $t \cdot K$, is defined as the unique convex body with centroid at the origin such that

$$S(t \cdot K, \cdot) = t \, S(K, \cdot).$$

Hence $t \cdot K = t^{1/(n-1)} K$, if K has its centroid at the origin. A convex body is origin-symmetric if and only if its surface area measure is an even measure and its centroid is at the origin. Note that for $K \in \mathcal{K}_0^n$, the *Blaschke symmetral* $\frac{1}{2} \cdot (K \# (-K))$ is an origin-symmetric convex body. Also note that $\langle \mathcal{K}_c^n, \# \rangle$ is an abelian semigroup.

For $K \in \mathcal{K}^n$ which contains the origin in its interior, the *polar body*, K^*, of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \quad \text{for every } \ y \in K\}.$$

For a normed space $E = (\mathbb{R}^n, \|\cdot\|)$, the dual space is $E^* = (\mathbb{R}^n, \|\cdot\|_*)$, where $\|\cdot\|_*$ is given for $v \in \mathbb{R}^n$ by

$$\|v\|_* = \sup\{x \cdot v : \|x\| \leqslant 1\}.$$

If B is the unit ball of E, that is, $B = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$, then its polar body, B^*, is the unit ball of E^*.

We require some facts about the projection operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$, which can be found in [17]. It is a simple consequence of the definition of Π that

$$\Pi(K \# L) = \Pi K + \Pi L \tag{6}$$

for $K, L \in \mathcal{K}_0^n$. Note that for $K \in \mathcal{K}_0^n$, we have

$$\Pi\left(\tfrac{1}{2} \cdot (K \# (-K))\right) = \Pi K. \tag{7}$$

The projection operator has strong contravariance and invariance properties: for all $\phi \in \mathrm{GL}(n)$ and translations τ, we have

$$\Pi(\phi K) = |\det \phi|\, \phi^{-t}\, \Pi\, K \quad \text{and} \quad \Pi(\tau K) = \Pi\, K \tag{8}$$

for all $K \in \mathcal{K}_0^n$. Further, Π is continuous on $\mathcal{K}_0^n$ and injective on $\mathcal{K}_c^n$. If $\mathcal{Z}^n$ denotes the range of Π, the inverse operator $\Pi^{-1} : \mathcal{Z}^n \to \mathcal{K}_c^n$ is also continuous.

The proofs of Theorems 1 and 2 make essential use of a classification result of convex-body-valued valuations established in [36]. To state the result, we need the following definitions. Let $\mathcal{P}_0^n$ denote the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors. The *moment body*, $\mathrm{M}\, P$, of P is defined by

$$h(\mathrm{M}\, P, v) = \int_P |v \cdot x|\, dx, \;\; v \in \mathbb{R}^n.$$

We say that an operator $\mathrm{Z} : \mathcal{P}_0^n \to \mathcal{K}^n$ is $\mathrm{GL}(n)$ *contravariant of weight* $p \in \mathbb{R}$, if

$$\mathrm{Z}(\phi P) = |\det \phi|^p\, \phi^{-t}\, \mathrm{Z}\, P$$

for all $P \in \mathcal{P}_0^n$ and $\phi \in \mathrm{GL}(n)$.

Theorem 3 ([36])**.** *An operator* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}_c^n, + \rangle$, *where* $n \geqslant 3$, *is a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* p *if and only if there is a constant* $c \geqslant 0$ *such that*

$$\mathrm{Z}\, P = \begin{cases} c\, \mathrm{M}\, P^* & \text{for } p = -1 \\ c(P^* + (-P)^*) & \text{for } p = 0 \\ c\, \Pi\, P & \text{for } p = 1 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^n$.

For $n = 2$, there are additional convex-body-valued valuations (see [36]). Also note that if we replace $\mathrm{GL}(n)$ contravariance by $\mathrm{SO}(n)$ covariance, there is a much larger class of valuations (see, for example, [52]).

2 Background material on Sobolev spaces

For $p \geqslant 1$ and a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, let

$$|f|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx \right)^{1/p}.$$

A measurable function f is in $L^p(\mathbb{R}^n)$ if $|f|_p < \infty$. A function $f \in L^1(\mathbb{R}^n)$ has L^1 *weak derivative*, if there exists a measurable function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\nabla f \in L^1(\mathbb{R}^n)$ (that is, $|\nabla f| \in L^1(\mathbb{R}^n)$) and

$$\int_{\mathbb{R}^n} \nu(x) \cdot \nabla f(x)\, dx = -\int_{\mathbb{R}^n} f(x) \nabla \cdot \nu(x)\, dx$$

for every compactly supported smooth vector field $\nu(x) : \mathbb{R}^n \to \mathbb{R}^n$, where we use the notation $\nabla \cdot \nu = \frac{\partial \nu}{\partial x_1} + \cdots + \frac{\partial \nu}{\partial x_n}$. The function ∇f is called the *weak gradient* of f and the L^1 norm of $|\nabla f|$ is denoted by $|\nabla f|_1$.

An operator $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is continuous, if for every sequence $f_k \in W^{1,1}(\mathbb{R}^n)$ with $f_k \to f$ as $k \to \infty$ in $W^{1,1}(\mathbb{R}^n)$, we have $\delta(\mathrm{z}(f_k), \mathrm{z}(f)) \to 0$ as $k \to \infty$. Here $f_k \to f$ as $k \to \infty$ in $W^{1,1}(\mathbb{R}^n)$ if $|f_k - f|_1 \to 0$ and $|\nabla(f_k - f)|_1 \to 0$ as $k \to \infty$. An operator $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is called trivial, if $\mathrm{z}(f) = \{0\}$ for all $f \in W^{1,1}(\mathbb{R}^n)$. It is called $\mathrm{GL}(n)$ *covariant of weight* $p \in \mathbb{R}$, if

$$\mathrm{z}(f \circ \phi^{-1}) = |\det \phi|^p \, \phi \, \mathrm{z}(f)$$

for all $f \in W^{1,1}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$. It is called $\mathrm{GL}(n)$ *contravariant of weight* $p \in \mathbb{R}$, if

$$\mathrm{z}(f \circ \phi^{-1}) = |\det \phi|^p \, \phi^{-t} \, \mathrm{z}(f)$$

for all $f \in W^{1,1}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$. It is called *homogeneous of degree* $q \in \mathbb{R}$, if

$$\mathrm{z}(sf) = |s|^q \, \mathrm{z}(f)$$

for all $f \in W^{1,1}(\mathbb{R}^n)$ and $s \in \mathbb{R}$. If an operator $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is homogeneous of degree q and non-trivial, then setting $s = 0$ in the definition of homogeneity gives $q \geqslant 0$. If $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is continuous and homogeneous of degree 0, then $\mathrm{z}(f) = \mathrm{z}(0)$ for all $f \in W^{1,1}(\mathbb{R}^n)$. If z is in addition $\mathrm{GL}(n)$ co- or contravariant, then we obtain that z is trivial. In particular, we have

$$\mathrm{z}(0) = \{0\} \tag{9}$$

for all continuous, homogeneous and $\mathrm{GL}(n)$ co- or contravariant $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$.

For $f, g \in W^{1,1}(\mathbb{R}^n)$, $f \vee g, f \wedge g \in W^{1,1}(\mathbb{R}^n)$ and for almost every $x \in \mathbb{R}^n$,

$$\nabla(f \vee g)(x) = \begin{cases} \nabla f(x) & \text{when } f(x) > g(x) \\ \nabla g(x) & \text{when } f(x) < g(x) \\ \nabla f(x) = \nabla g(x) & \text{when } f(x) = g(x) \end{cases} \tag{10}$$

and

$$\nabla(f \wedge g) = \begin{cases} \nabla f(x) & \text{when } f(x) < g(x) \\ \nabla g(x) & \text{when } f(x) > g(x) \\ \nabla f(x) = \nabla g(x) & \text{when } f(x) = g(x) \end{cases} \tag{11}$$

(see, for example, [32]). Hence $(W^{1,1}(\mathbb{R}^n), \vee, \wedge)$ is a lattice.

Let $L^{1,1}(\mathbb{R}^n) \subset W^{1,1}(\mathbb{R}^n)$ denote the space of *piecewise affine functions* on $\mathbb{R}^n$, where a function $\ell : \mathbb{R}^n \to \mathbb{R}$ is called piecewise affine, if ℓ is continuous and there are finitely many n-dimensional simplices $S_1, \ldots, S_m \subset \mathbb{R}^n$ with pairwise disjoint interiors such that the restriction of ℓ to each S_i is affine and $\ell = 0$ outside $S_1 \cup \cdots \cup S_m$. Note that the simplices $S_1, \ldots, S_m$ are a triangulation of the support of ℓ. Further, note that if V is the set of vertices of this triangulation, then V and the values $\ell(v)$ for $v \in V$ completely determine ℓ. Piecewise affine functions lie dense in $W^{1,1}(\mathbb{R}^n)$ (see, for example, [31]).

For $P \in \mathcal{P}_0^n$, define the piecewise affine function ℓ_P by requiring that $\ell_P(0) = 1$, that $\ell_P(x) = 0$ for $x \notin P$, and that ℓ_P is affine on each simplex with apex at the origin and base equal to a facet of P. Define $P^{1,1}(\mathbb{R}^n) \subset L^{1,1}(\mathbb{R}^n)$ as the set of all ℓ_P for $P \in \mathcal{P}_0^n$. Note that for $\phi \in \mathrm{GL}(n)$,

$$\ell_{\phi P} = \ell_P \circ \phi^{-1}. \tag{12}$$

We remark that multiples and translates of $\ell_P \in P^{1,1}(\mathbb{R}^n)$ correspond to linear elements within the theory of finite elements.

3 The operators $f \mapsto \langle f \rangle$ and $f \mapsto \Pi \langle f \rangle$

The operator $f \mapsto \langle f \rangle$ has strong covariance and invariance properties (see [45] and also [40]). In particular,

$$\langle s\, f \rangle = |s| \cdot \langle f \rangle, \quad \langle f \circ \phi^{-1} \rangle = \phi \langle f \rangle, \quad \langle f \circ \tau^{-1} \rangle = \langle f \rangle \tag{13}$$

for all $s \in \mathbb{R}$, $\phi \in \mathrm{GL}(n)$ and for all translations τ.

Lemma 1. *The operator* $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$*, defined by* $\mathrm{z}(f) = c\, \Pi \langle f \rangle$ *with* $c \geqslant 0$*, is a continuous affinely contravariant valuation.*

Proof. Using (10) and (11), we obtain from (4) and (5) that z is a valuation. By (13) and (8), we see that z is affinely contravariant. Suppose that $f_k \to f$ as $k \to \infty$ in $W^{1,1}(\mathbb{R}^n)$. Then for $u \in S^{n-1}$ we have by (4), the reverse triangle inequality and the Cauchy-Schwarz inequality,

$$|h(\mathrm{z}(f_k), u) - h(\mathrm{z}(f), u)| \leqslant \frac{c}{2} \int_{\mathbb{R}^n} |u \cdot \nabla (f_k - f)(x)|\, dx \leqslant \frac{c}{2} \int_{\mathbb{R}^n} |\nabla (f_k - f)(x)|\, dx.$$

Therefore we obtain $\delta(\mathrm{z}(f_k), \mathrm{z}(f)) \to 0$ as $k \to 0$ and thus z is continuous. □

Lemma 2. *The operator* $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, \# \rangle$*, defined by* $\mathrm{z}(f) = c \langle f \rangle$ *with* $c \geqslant 0$*, is a continuous affinely covariant valuation.*

Proof. Since the inverse projection operator Π^{-1} is continuous, Lemma 1 implies that z is continuous. By (13), z is affinely covariant. Since by Lemma 1 for all $f, g \in W^{1,1}(\mathbb{R}^n)$,

$$\Pi\, \mathrm{z}(f) + \Pi\, \mathrm{z}(g) = \Pi\, \mathrm{z}(f \vee g) + \Pi\, \mathrm{z}(f \wedge g),$$

we obtain by (6) that

$$\Pi \left(\mathrm{z}(f) \# \mathrm{z}(g) \right) = \Pi \left(\mathrm{z}(f \vee g) \# \mathrm{z}(f \wedge g) \right).$$

Applying Π^{-1} gives

$$\mathrm{z}(f) \# \mathrm{z}(g) = \mathrm{z}(f \vee g) \# \mathrm{z}(f \wedge g)$$

for all $f, g \in W^{1,1}(\mathbb{R}^n)$. Thus $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, \# \rangle$ is a valuation. □

Lemma 3. *For* $P \in \mathcal{P}_0^n$, $\langle \ell_P \rangle = \frac{1}{2} \cdot (P \,\#\, (-P))$.

Proof. By definition, $\langle \ell_P \rangle = \frac{1}{2} \cdot (P \,\#\, (-P))$ if for every even $g \in C(\mathbb{R}^n)$ that is homogeneous of degree 1,

$$\frac{1}{n} \int_{S^{n-1}} g(u)\, dS(\tfrac{1}{2} \cdot (P \,\#\, (-P)), u) = \int_{\mathbb{R}^n} g(-\nabla \ell_P)\, dx.$$

Let P have facets $F_1, \ldots, F_m$. For the facet F_i, let u_i be its unit outer normal vector and T_i the convex hull of F_i and the origin. Since for $x \in T_i$

$$\ell_P(x) = -\frac{u_i}{h(P, u_i)} \cdot x + 1$$

and

$$\nabla \ell_P(x) = -\frac{u_i}{h(P, u_i)},$$

we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^n} g(-\nabla \ell_P)\, dx &= \sum_{i=1}^m \int_{T_i} g(-\nabla \ell_P(x))\, dx \\
&= \sum_{i=1}^m g(u_i)\, \frac{V_n(T_i)}{h(P, u_i)} \\
&= \frac{1}{n} \sum_{i=1}^m g(u_i)\, V_{n-1}(F_i) \\
&= \frac{1}{n} \int_{S^{n-1}} g(u)\, dS(P, u) \\
&= \frac{1}{n} \int_{S^{n-1}} g(u)\, dS(\tfrac{1}{2} \cdot (P \,\#\, (-P)), u).
\end{aligned}$$

Thus $\langle \ell_P \rangle = \frac{1}{2} \cdot (P \,\#\, (-P))$. □

4 Proof of Theorem 2

In Lemma 1, it was shown that $f \mapsto c\,\Pi \langle f \rangle$ is for $c \geqslant 0$ a continuous affinely contravariant valuation. Suppose that z is a continuous affinely contravariant valuation. The following lemmas establish that there is a constant $c \geqslant 0$ such that $\mathrm{z}(f) = c\,\Pi \langle f \rangle$ for all $f \in W^{1,1}(\mathbb{R}^n)$.

Lemma 4. *If* $\mathrm{z} : P^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ *is continuous, non-trivial, and* $\mathrm{GL}(n)$ *contravariant of weight* p*, then* $p \geqslant 1$.

Proof. For $a > 0$ and $0 < \varepsilon < 1$, let $\phi_a \in \mathrm{GL}(n)$ map e_1 to $a\, e_1$ and e_i to $a^\varepsilon e_i$ for $i = 2, \ldots, n$. Using (12), we obtain for $P \in \mathcal{P}_0^n$ that $|\ell_{\phi_a P}|_1 = |\det \phi_a|\, |\ell_P|_1$ and

$$|\nabla \ell_{\phi_a P}|_1 = |\det \phi_a| \int_{\mathbb{R}^n} |\phi_a^{-t}\, \nabla \ell_P(x)|\, dx \leqslant |\det \phi_a|\, |\nabla \ell_P|_1 \max\{|\phi_a^{-t} u| : u \in S^{n-1}\}.$$

Hence $|\ell_{\phi_a P}|_1 = O(a^{1+(n-1)\varepsilon})$ and $|\nabla \ell_{\phi_a P}|_1 = O(a^{(n-1)\varepsilon})$ as $a \to 0$. Consequently, $\ell_{\phi_a P} \to 0$ in $W^{1,1}(\mathbb{R}^n)$ as $a \to 0$. Since z is GL(n) contravariant of weight p, we obtain by (12) that

$$\mathrm{z}(\ell_{\phi_a P}) = a^{(1+(n-1)\varepsilon)p} \phi_a^{-t}\, \mathrm{z}(\ell_P).$$

Thus the first coordinates of points from $\mathrm{z}(\ell_P)$ are multiplied by $a^{(1+(n-1)\varepsilon)p-1}$. Since this happens for all $P \in \mathcal{P}_0^n$ and z is continuous, we conclude that $p \geqslant 1/(1+(n-1)\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we obtain $p \geqslant 1$. □

Lemma 5. *If* $\mathrm{z} : P^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$, *where* $n \geqslant 3$, *is a continuous affinely contravariant valuation, then there is a constant* $c \geqslant 0$ *such that*

$$\mathrm{z}(f) = c\, \Pi \langle f \rangle$$

for every $f \in P^{1,1}(\mathbb{R}^n)$.

Proof. Define the operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}_c^n, + \rangle$ by setting

$$\mathrm{Z}\, P = \mathrm{z}(\ell_P).$$

If $\ell_P, \ell_Q \in P^{1,1}(\mathbb{R}^n)$ are such that $\ell_P \vee \ell_Q \in P^{1,1}(\mathbb{R}^n)$, then $\ell_P \vee \ell_Q = \ell_{P \cup Q}$ and $\ell_P \wedge \ell_Q = \ell_{P \cap Q}$. Since z is a valuation on $P^{1,1}(\mathbb{R}^n)$, it follows for $P, Q, P \cup Q \in \mathcal{P}_0^n$ that

$$\begin{aligned} \mathrm{Z}(P) + \mathrm{Z}(Q) &= \mathrm{z}(\ell_P) + \mathrm{z}(\ell_Q) \\ &= \mathrm{z}(\ell_P \vee \ell_Q) + \mathrm{z}(\ell_P \wedge \ell_Q) \\ &= \mathrm{Z}(P \cup Q) + \mathrm{Z}(P \cap Q). \end{aligned}$$

Thus $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}_c^n, + \rangle$ is a valuation.

By Lemma 4, the operator z is GL(n) contravariant of weight $p \geqslant 1$. Since for $\phi \in \mathrm{GL}(n)$

$$\mathrm{Z}(\phi P) = \mathrm{z}(\ell_{\phi P}) = \mathrm{z}(\ell_P \circ \phi^{-1}) = |\det \phi|^p\, \phi^{-t}\, \mathrm{z}(\ell_P) = |\det \phi|^p\, \phi^{-t}\, \mathrm{Z}\, P,$$

also Z is GL(n) contravariant of weight $p \geqslant 1$. Thus we obtain from Theorem 3 that there exists a constant $c \geqslant 0$ such that

$$\mathrm{z}(\ell_P) = c\, \Pi\, P$$

for all $\ell_P \in P^{1,1}(\mathbb{R}^n)$. The statement now follows from Lemma 3 and (7). □

Lemma 6. *If* $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$ *is a continuous, non-trivial, translation invariant valuation which is homogeneous of degree* q, *then* $q \geqslant 1$.

Proof. Let $P \in \mathcal{P}_0^n$ and $\varepsilon > 0$. Take translations $\tau_1, \dots, \tau_k$ such that the polytopes $\tau_i P$ are pairwise disjoint. Define

$$f_k = \frac{1}{k^{1+\varepsilon}} (\ell_{\tau_1 P} \vee \cdots \vee \ell_{\tau_k P}).$$

Then $|f_k|_1 = |\nabla f_k|_1 = O(k^{-\varepsilon})$ as $k \to \infty$. Hence $f_k \to 0$ as $k \to \infty$ in $W^{1,1}(\mathbb{R}^n)$. Since z is a translation invariant and homogeneous valuation, we obtain using (9) that

$$\mathrm{z}(f_k) = k\, k^{-q(1+\varepsilon)}\, \mathrm{z}(\ell_P).$$

Since z is continuous, it follows from (9) that $q \geqslant 1$. □

Lemma 7. *If* $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}^n_c, +\rangle$ *is a continuous, non-trivial, translation invariant valuation which is* $\mathrm{GL}(n)$ *contravariant of weight 1 and homogeneous of degree* q*, then* $q \leqslant 1$.

Proof. Let $P \in \mathcal{P}^n_0$ and $\alpha, \beta > 0$. Take translations $\tau_1, \dots, \tau_k$ such that the polytopes $\tau_i P$ are pairwise disjoint. Define

$$f_k = k^{\alpha}(\ell_{\tau_1(P/k^{\beta})} \vee \cdots \vee \ell_{\tau_k(P/k^{\beta})}).$$

Then $|f_k|_1 = O(k^{1+\alpha-n\beta})$ and $|\nabla f_k|_1 = O(k^{1+\alpha+\beta-n\beta})$ as $k \to \infty$. Hence for $\alpha < (n-1)\beta - 1$, we have $f_k \to 0$ as $k \to \infty$ in $W^{1,1}(\mathbb{R}^n)$. Since z is a translation invariant and homogeneous valuation, we obtain using (9) that

$$\mathrm{z}(f_k) = k\, k^{\alpha q} k^{-n\beta+\beta}\, \mathrm{z}(\ell_P).$$

Since z is continuous, it follows from (9) that $q \leqslant (-1 + (n-1)\beta)/\alpha$. Since this holds for all $\alpha < (n-1)\beta - 1$ and all β, we conclude that $q \leqslant 1$. □

Lemma 8. *Let* $\mathrm{z}_1, \mathrm{z}_2 : L^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}^n, +\rangle$ *be continuous, translation invariant valuations, which are homogeneous of the same degree. If* $\mathrm{z}_1(f) = \mathrm{z}_2(f)$ *for all* $f \in P^{1,1}(\mathbb{R}^n)$*, then*

$$\mathrm{z}_1(f) = \mathrm{z}_2(f) \tag{14}$$

for all $f \in L^{1,1}(\mathbb{R}^n)$.

Proof. Let z_1 and z_2 be homogeneous of degree q. As noted before, $q \geqslant 0$. If $q = 0$, then $\mathrm{z}_i(f) = \mathrm{z}_i(0)$ for all $f \in L^{1,1}(\mathbb{R}^n)$ and the statement of the lemma is true. Therefore we assume that z_1 and z_2 are homogeneous of degree $q > 0$ and have

$$\mathrm{z}_1(0) = \mathrm{z}_2(0) = \{0\}. \tag{15}$$

Since z_1 and z_2 are homogeneous valuations, we obtain using (15) that for $i = 1, 2$,

$$\mathrm{z}_i(f \vee 0) + \mathrm{z}_i(f \wedge 0) = \mathrm{z}_i(f) + \mathrm{z}_i(0) = \mathrm{z}_i(f)$$

and

$$\mathrm{z}_i(f \wedge 0) = \mathrm{z}_i(-((-f) \vee 0)) = \mathrm{z}_i((-f) \vee 0).$$

Thus it suffices to show that (14) holds for all $f \in L^{1,1}(\mathbb{R}^n)$ with $f \geqslant 0$.

Let such a function f be given and let f not vanish identically. Triangulate the support of f so that f is affine on each simplex of the triangulation. Let V be the (finite) set of vertices and $\mathcal{S}$ the set of n-dimensional simplices of this triangulation. Note that f is determined by the values $f(v)$ for $v \in V$ and that if $f(\bar{v}) > 0$ for some $\bar{v} \in V$, then by changing the value $f(\bar{v})$ we obtain again a function in $L^{1,1}(\mathbb{R}^n)$. Since z_1 and z_2 are continuous, it suffices to prove (14) for a function f where the values $f(v)$ are distinct for $v \in V$ with $f(v) > 0$.

First, we show that for such a function f there are $f_1, \dots, f_m \in L^{1,1}(\mathbb{R}^n)$ which are non-negative and concave on their supports such that

$$f = f_1 \vee \cdots \vee f_m. \tag{16}$$

Define the function f_i by setting $f_i(v) = f(v)$ on the vertices v of the simplex S_i of $\mathcal{S}$. Choose a polytope P_i such that $S_i \subset P_i$ and set $f_i(v) = 0$ on the vertices v of P_i. The function f_i determined by these data is concave on its support and piecewise linear. Moreover, if the polytopes P_i are chosen suitably small, (16) holds.

Using the inclusion-exclusion principle, we obtain from (16) that for $i = 1, 2$,

$$\mathrm{z}_i(f) = \mathrm{z}_i(f_1 \vee \cdots \vee f_m) = \sum_J (-1)^{|J|-1}\, \mathrm{z}_i(f_J)$$

where we sum over all non-empty $J \subset \{1, \dots, m\}$ and

$$f_J = f_{j_1} \wedge \cdots \wedge f_{j_k}$$

for $J = \{j_1, \dots, j_k\}$. Thus it suffices to prove (14) for non-negative $f \in L^{1,1}(\mathbb{R}^n)$ that are concave on their support.

For a given function $f \in L^{1,1}(\mathbb{R}^n)$, let $F \subset \mathbb{R}^{n+1}$ be the compact polytope bounded by the graph of f and the hyperplane $\{x_{n+1} = 0\}$. We call F *singular* if F has n facet hyperplanes that intersect in a line L parallel to $\{x_{n+1} = 0\}$ but not contained in $\{x_{n+1} = 0\}$. Since z_1 and z_2 are continuous, it suffices to show (14) for $f \in L^{1,1}(\mathbb{R}^n)$ such that F is not singular. So we assume for the rest of the proof that f has this property.

Let such a function f be given. Let $\bar{p}$ be the vertex of F with the largest x_{n+1} coordinate. We use induction on the number m of facet hyperplanes of F that are not passing through $\bar{p}$. If $m = 1$, then a translate of f is in $P^{1,1}(\mathbb{R}^n)$. Since z_1 and z_2 are translation invariant and homogeneous, (14) is true. Suppose (14) is true for all $f \in L^{1,1}(\mathbb{R}^n)$ such that F has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$. We show that (14) then also holds for all $f \in L^{1,1}(\mathbb{R}^n)$ with m such hyperplanes.

So let F have m such hyperplanes. Let $p_0 = (x_0, f(x_0))$ be a vertex of F with minimal non-negative x_{n+1}-coordinate. Let $H_1, \dots, H_j$ be the facet hyperplanes of F through p_0 which do not contain $\bar{p}$. There is at least one such hyperplane. Define $\bar{F}$ as the polytope bounded by the intersection of all facet hyperplanes of F with the exception of $H_1, \dots, H_j$. Since F has no edges parallel to $\{x_{n+1} = 0\}$ but not contained in $\{x_{n+1} = 0\}$, the polytope $\bar{F}$ is bounded and the piecewise affine function $\bar{f}$ corresponding to $\bar{F}$ is in $L^{1,1}(\mathbb{R}^n)$. Note that $\bar{F}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$. Let $\bar{H}_1, \dots, \bar{H}_i$ be the facet hyperplanes of $\bar{F}$ that contain p_0. Choose suitable hyperplanes $\bar{H}_{i+1}, \dots, \bar{H}_k$ containing p_0 so that the hyperplanes $\bar{H}_1, \dots, \bar{H}_k$ and $\{x_{n+1} = 0\}$ bound a pyramid with apex at p_0 that is contained in $\bar{F}$, has x_0 in its base and has $\bar{H}_1, \dots, \bar{H}_i$ among its facet hyperplanes. Define ℓ as the piecewise affine function determined by this pyramid and note that a suitable translate of ℓ is in $P^{1,1}(\mathbb{R}^n)$. Set $\bar{\ell} = f \wedge \ell \in L^{1,1}(\mathbb{R}^n)$. The polytope determined by $\bar{\ell}$ is a pyramid since it is bounded by $\{x_{n+1} = 0\}$ and hyperplanes containing p_0. Therefore a suitable translate of $\bar{\ell}$ is in $P^{1,1}(\mathbb{R}^n)$. Hence translates of $\bar{\ell}$ and ℓ are in $P^{1,1}(\mathbb{R}^n)$, the polytope $\bar{F}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$, and

$$f \vee \ell = \bar{f} \quad \text{and} \quad f \wedge \ell = \bar{\ell}.$$

Since z is a valuation, we obtain for $i = 1, 2$ that

$$\mathrm{z}_i(f) + \mathrm{z}_i(\ell) = \mathrm{z}_i(\bar{f}) + \mathrm{z}_i(\bar{\ell}).$$

Thus by induction (14) holds for all $f \in L^{1,1}(\mathbb{R}^n)$ with m facet hyperplanes not containing $\bar{p}$. This completes the proof of the lemma. □

5 Proof of Theorem 1

Suppose that $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, \# \rangle$ is a continuous affinely covariant valuation. Then for all $f, g \in W^{1,1}(\mathbb{R}^n)$,

$$\mathrm{z}(f) \# \mathrm{z}(g) = \mathrm{z}(f \vee g) \# \mathrm{z}(f \wedge g).$$

Hence, applying Π, we obtain by (6) that

$$\Pi\, \mathrm{z}(f) + \Pi\, \mathrm{z}(g) = \Pi\, \mathrm{z}(f \vee g) + \Pi\, \mathrm{z}(f \wedge g)$$

for all $f, g \in W^{1,1}(\mathbb{R}^n)$, that is, $\Pi \circ \mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$ is a valuation. Since z is affinely covariant, (8) implies that $\Pi \circ \mathrm{z}$ is affinely contravariant. Since z and Π are continuous, also $\Pi \circ \mathrm{z}$ is continuous. Thus by Theorem 2, there is a constant $\tilde{c} \geqslant 0$ such that

$$\Pi\, \mathrm{z}(f) = \tilde{c}\, \Pi \langle f \rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$. Since Π is injective on $\mathcal{K}_c^n$, we obtain that $\mathrm{z}(f) = c\langle f \rangle$ for all $f \in W^{1,1}(\mathbb{R}^n)$ for some $c \in \mathbb{R}$. Combined with Lemma 2, this completes the proof of the theorem.

Acknowledgements

The author is most grateful to the referees for their critique of the original draft of this paper and for their valuable remarks.

References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. (2) **149** (1999), 977–1005.

[2] S. Alesker, *Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture*, Geom. Funct. Anal. **11** (2001), 244–272.

[3] S. Alesker, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom. **63** (2003), 63–95.

[4] S. Alesker, *The multiplicative structure on continuous polynomial valuations*, Geom. Funct. Anal. **14** (2004), 1–26.

[5] S. Alesker, *Valuations on convex sets, non-commutative determinants, and pluripotential theory*, Adv. Math. **195** (2005), 561–595.

[6] S. Alesker, *Theory of valuations of manifolds, II*, Adv. Math. **207** (2006), 420–454.

[7] S. Alesker, *Theory of valuations on manifolds: a survey*, Geom. Funct. Anal. **17** (2007), 1321–1341.

[8] S. Alesker and J. H. G. Fu, *Theory of valuations on manifolds. III. Multiplicative structure in the general case*, Trans. Amer. Math. Soc. **360** (2008), 1951–1981.

[9] A. Bernig, *Valuations with Crofton formula and Finsler geometry*, Adv. Math. **210** (2007), 733–753.

[10] A. Bernig, *A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line*, Comment. Math. Helv. **84** (2009), 1–19.

[11] A. Bernig and L. Bröcker, *Valuations on manifolds and Rumin cohomology*, J. Differential Geom. **75** (2007), 433–457.

[12] S. Campi, A. Colesanti, and P. Gronchi, *Blaschke-decomposable convex bodies*, Israel J. Math. **105** (1998), 185–195.

[13] A. Cianchi, E. Lutwak, D. Yang, and G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*, Calc. Var. Partial Differential Equations **36** (2009), 419–436.

[14] D. Cordero-Erausquin, B. Nazaret, and C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math. **182** (2004), 307–332.

[15] H. Federer and W. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520.

[16] J. H. G. Fu, *Structure of the unitary valuation algebra*, J. Differential Geom. **72** (2006), 509–533.

[17] R. Gardner, *Geometric tomography*, second ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.

[18] E. Grinberg and G. Zhang, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc. (3) **78** (1999), 77–115.

[19] P. M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 336, Springer, Berlin, 2007.

[20] C. Haberl, *Star body valued valuations*, Indiana Univ. Math. J. **58** (2009), 2253–2276.

[21] C. Haberl, *Blaschke valuations*, Amer. J. Math., in press.

[22] C. Haberl, *Minkowski valuations intertwining the special linear group*, preprint.

[23] C. Haberl and M. Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not. **10548** (2006), 1–29.

[24] C. Haberl and F. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. **257** (2009), 641–658.

[25] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.

[26] M. Kiderlen, *Blaschke- and Minkowski-endomorphisms of convex bodies*, Trans. Amer. Math. Soc. **358** (2006), 5539–5564.

[27] D. A. Klain, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), 80–101.

[28] D. A. Klain, *Invariant valuations on star-shaped sets*, Adv. Math. **125** (1997), 95–113.

[29] D. A. Klain, *Covering shadows with a smaller volume*, Adv. Math. 224 (2010), 601–619.

[30] D. A. Klain and G.-C. Rota, *Introduction to geometric probability*, Cambridge University Press, Cambridge, 1997.

[31] G. Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009.

[32] E. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[33] M. Ludwig, *Projection bodies and valuations*, Adv. Math. **172** (2002), 158–168.

[34] M. Ludwig, *Ellipsoids and matrix valued valuations*, Duke Math. J. **119** (2003), 159–188.

[35] M. Ludwig, *Intersection bodies and valuations*, Amer. J. Math. **128** (2006), 1409–1428.

[36] M. Ludwig, *Minkowski areas and valuations*, J. Differential Geom., in press.

[37] M. Ludwig, *Covariance matrices and valuations*, preprint.

[38] M. Ludwig and M. Reitzner, *A characterization of affine surface area*, Adv. Math. **147** (1999), 138–172.

[39] M. Ludwig and M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. of Math. **172** (2010), 1223–1271.

[40] M. Ludwig, J. Xiao, and G. Zhang, *Sharp convex Lorentz-Sobolev inequalities*, Math. Ann., in press.

[41] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.

[42] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. **60** (1990), 365–391.

[43] E. Lutwak, *Extended affine surface area*, Adv. Math. **85** (1991), 39–68.

[44] E. Lutwak, D. Yang, and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), 17–38.

[45] E. Lutwak, D. Yang, and G. Zhang, *Optimal Sobolev norms and the L^p Minkowski problem*, Int. Math. Res. Not. **62987** (2006), 1–10.

[46] V. G. Maz′ya, *Classes of domains and imbedding theorems for function spaces*, Soviet Math. Dokl. **1** (1960), 882–885.

[47] P. McMullen, *Valuations and dissections*, Handbook of Convex Geometry, Vol. B (P.M. Gruber and J.M. Wills, eds.), North-Holland, Amsterdam, 1993, 933–990.

[48] P. McMullen and R. Schneider, *Valuations on convex bodies*, Convexity and its applications (P.M. Gruber and J.M. Wills, eds.), Birkhäuser, 1983, 170–247.

[49] V. D. Milman and G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Mathematics, vol. 1200, Springer-Verlag, Berlin, 1986, With an appendix by M. Gromov.

[50] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.

[51] R. Schneider and F. Schuster, *Rotation equivariant Minkowski valuations*, Int. Math. Res. Not. **72894** (2006), 1–20.

[52] F. Schuster, *Convolutions and multiplier transformations of convex bodies*, Trans. Amer. Math. Soc. **359** (2007), 5567–5591.

[53] F. Schuster, *Valuations and Busemann-Petty type problems*, Adv. Math. **219** (2008), 344–368.

[54] F. Schuster and T. Wannerer, *GL(n) contravariant Minkowski valuations*, Trans. Amer. Math. Soc., in press.

[55] A. C. Thompson, *Minkowski geometry*, Cambridge University Press, Cambridge, 1996.

[56] A. Tsang, *Valuations on L^p spaces*, Int. Math. Res. Not., in press.

[57] A. Tsang, *Minkowski valuations on L^p spaces*, preprint.

[58] T. Wannerer, *GL(n) equivariant Minkowski valuations*, Indiana Univ. Math. J., in press.

[59] J. Xiao, *The sharp Sobolev and isoperimetric inequalities split twice*, Adv. Math. **211** (2007), 417–435.

[60] G. Zhang, *The affine Sobolev inequality*, J. Differential Geom. **53** (1999), 183–202.

Institut für Diskrete Mathematik und Geometrie, TU Wien,
Wiedner Hauptstr. 8-10/104, 1040 Wien, Austria

and

Department of Mathematics, Polytechnic Institute of New York University,
6 MetroTech Center, Brooklyn, NY 11201, U.S.A.

E-mail: mludwig@poly.edu