# NOTES ON RIEMANN-FINSLER GEOMETRY

VAN ABEL

## Part 1. [1]NOTE OF CHAPTER 1, Finsler Metric

### 1. Finsler Metrics I

#### 1.1. Minkowski Norm.

1.1.1. *homogenous function.* Let $\mathscr{U} \subset \mathbf{R^n}$ is an open domain. $f : \mathscr{U} \to \mathbf{R}$ is positively homo.func. of deg $r \Leftrightarrow f(\lambda u) = \lambda^r f(u), \lambda > 0$.

**Theorem 1.1** (Euler theorem). *A $C^1$ function $f(u)$ on $\mathscr{U}$ is $(r)p-$ homo-genous(It means that f is a homogenous function at p of order r.) if and only if*

$$\frac{\partial f}{\partial u^i} u^i = r f(u).$$

*Proof.* Assume that

$$f(\lambda u) = \lambda^r f(u), \quad \lambda > 0.$$

diff. w.r.t. $\lambda$

$$\frac{\partial f}{\partial u^i}(\lambda u) u^i = r\lambda^{r-1} f(u).$$

Let $\lambda = 1$, then

$$\frac{\partial f}{\partial u^i} u^i = r f(u).$$

Conversely, assume that $\frac{\partial f}{\partial u^i} u^i = r f(u)$. Then

$$\begin{aligned} r f(\lambda u) &= \frac{\partial f}{\partial u^i}(\lambda u) \lambda u^i \\ &= \frac{\partial f(\lambda u)}{\partial \lambda} \lambda. \end{aligned}$$

If we set $h(\lambda) = f(\lambda u)$, for each u. Then

$$\frac{\mathrm{d} \frac{h(\lambda)}{\lambda^r}}{\mathrm{d}\lambda} = \frac{(h'(\lambda))\lambda - r h(\lambda)}{\lambda^{r+1}} = 0$$

Finally, we get

$$\frac{h(\lambda)}{\lambda^r} = const. = h(1) = f(u).$$

□

Differential both side of $f(\lambda u) = \lambda^r f(u)$ with $u_i$, we have

[1]First Time of Lecture, Thursday, March 11, 2010

**Proposition 1.1.** *A $C^2$ function $f(u)$ on $\mathscr{U}$ is $(r)p-$ homogenous $\Rightarrow \frac{\partial f}{\partial u^i}$ is $(r-1)p-$homogenous.*

### 1.2. **Length Structure and Volume Form.**

1.2.1. *The Length of Curve.* Give a functional $F : TM \to \mathbf{R}^+$
Define the length of $C$ w.r.t $F$ :

$$\mathcal{L}(C) := \int_a^b F(x(t), \dot{x}(t))\mathrm{d}t$$

Suppose a parameter change $t = \phi(s), \mathrm{d}\phi/\mathrm{d}s > 0$. $\mathcal{L}$ is independent of the choice of parameter means that

$$\int_a^b F\left(x, \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\right)\mathrm{d}t = \int_a^b F\left(x, \frac{\mathrm{d}x}{\mathrm{d}s}\right)\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t.$$

So, by the continuous of $F$, we have

$$F\left(x, \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\right) = F\left(x, \frac{\mathrm{d}x}{\mathrm{d}s}\right)\frac{\mathrm{d}s}{\mathrm{d}t}.$$

This is $F(x, \lambda y) = \lambda F(x, y), \lambda > 0$.

**Proposition 1.2.** *The length $\mathcal{L}(C)$ is independent of the choice of parametric if and only if*

$$F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0.$$

1.2.2. *Minkowski Norm.*

**Definition 1.1.** Suppose $V$ is a vector space on $\mathbf{R}$. A functional $F : V \to \mathbf{R}^+$ is called a *Minkowski Norm* on $V$. If the following conditions satisfied:

- $F$ is smooth on $V \setminus \{0\}$,
- $F(\lambda y) = \lambda F(y)$, $\lambda > 0$,
- The bilinear function defined as following:

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]_{s=t=0}.$$

  is an inner product.

Pair $(V, F)$ called a *Minkowski Space*.

1.2.3. *Convexity.* Suppose $u, v \in V \setminus \{0\}$, $\bar{u} = u/F(u)$, $\bar{v} = v/F(v)$, Assume $\bar{u} \neq \bar{v}$, define $\phi(t) = F^2(t\bar{u} + (1-t)\bar{v})$. Then

$$\begin{aligned}\phi'(t) &= [F^2]_{y^i}(t\bar{u} + (1-t)\bar{v})(\bar{u}^i - \bar{v}^i),\\ \phi''(t) &= [F^2]_{y^i y^j}(t\bar{u} + (1-t)\bar{v})(\bar{u}^i - \bar{v}^i)(\bar{u}^j - \bar{v}^j)\\ &= 2g_y(\bar{u} - \bar{v}, \bar{u} - \bar{v}) > 0\end{aligned}$$

Note that $\phi(0) = \phi(1) = 1$, by a theorem of elementary mathematical analysis, we can derive $\phi(t) \leq 1, 0 \leq t \leq 1$.

Now put $t = F(u)/(F(u) + F(v)) \in [0, 1]$, we get

$$\begin{aligned}\phi(t) &= F^2\left(\frac{F(u)}{F(u)+F(v)}\bar{u} + \frac{F(v)}{F(u)+F(v)}\bar{v}\right)\\ &= F^2\left(\frac{u}{F(u)+F(v)} + \frac{v}{F(u)+F(v)}\right)\\ &= \left[\frac{1}{F(u)+F(v)}F(u+v)\right]^2 < 1\end{aligned}$$

As $F \geq 0$, the above inequality means that

$$F(u+v) < F(u) + F(v).$$

If $\bar{u} = \bar{v}$, then

$$\begin{aligned}F(u+v) &= F(F(u)\bar{u} + F(v)\bar{v})\\ &= F((F(u)+F(v))\bar{u})\\ &= (F(u)+F(v))F(u/F(u))\\ &= F(u)+F(v).\end{aligned}$$

So, the inequality $F(u + v) \leq F(u) + F(v)$ holds for all $u, v \in V$. This implies that $F$ is convex on $V$.

1.2.4. *Indicatrix.*

**Definition 1.2.** Suppose $(V, F)$ is a Minkowski space, we define $S_F := \{y \in V | F(y) = 1\}$ is the *Indicatrix* of $V$.

**Theorem 1.2** (Okuba's Method). *Given a Minkowski space* $(V, F)$*.* $f(y) = 0$ *be the equation of the indictrix of* $S_F$*, then* $F(y)$ *satisfy* $f(y/F(y)) = 0$*. Moreover, if for any* $y \in S_F := \{y | f(y) = 0\}$*, we have* $F(y) = 1$*, then* $F(x, y)(> 0)$ *is unique as a positive homogenous function of rank* 1 *with respect to* $y$*.*

*Proof.* In fact, $F(y/F(y)) = 1$, so $y/F(y) \in S_F$, thus $f(y/F(y)) = 0$.

Moreover, if $F$ and $\bar{F}$ both be a homogenous function of rank 1 with respect to $y$. For any $y \neq 0$, there is a non-zero $\lambda$, such that $\lambda y \in S_F$, ie., $f(\lambda y) = 0$. By the condition that $F$ and $\bar{F}$ must satisfied, $\lambda F(y) = F(\lambda y) = 1 = \bar{F}(\lambda y) = \lambda \bar{F}(y)$, thus $F(y) = \bar{F}(y)$. □

The important of the method is that the indicatrix is not only a subspace of $V$, but also can reflect some intrinsic properties of the metric space itself.

*Example* 1.1. In $\mathbf{R^2}$, let

$$S_F : (y^1)^2 + (y^2)^2 = 3\sqrt{(y^1)^2 + (y^2)^2} + y^1$$

is the indicatrix of $F$ on $V$. try to get $F$.

*Proof.* By Okuba's method

$$F(y) = \frac{(y^1)^2 + (y^2)^2}{3\sqrt{(y^1)^2 + (y^2)^2} + y^1},$$

In fact, It's only need to verify that $F$ Satisfy the equation

$$f(y/F(y)) = 0.$$

This is easy to verify, as if $y \in S_F$, then $F(y) = 1$, and $f(y/F(y)) = f(y) = 0$. □

1.2.5. *Hessian Matrix.* Suppose $(V, F)$ is a Minkowski space, $\{b_i\} \subset V$ is a base. $y = y^i b_i$, $F(y) = F(y^i b_i) \overset{\text{def}}{=} F(y^i)$. Define

$$g_{ij} := g_y(b_i, b_j) = \frac{1}{2}\left[F^2\right]_{y^i y^j}.$$

Then, by theorem 1.1 and proposition 1.1 we have

$$F^2(y) = g_{ij} y^i y^j, \quad F(y) = \sqrt{g_{ij} y^i y^j}.$$

### 1.3. **Some Important Examples.**

1.3.1. *Euclidean Norm.* Suppose $(V, \langle, \rangle)$ is a Euclidean space. $\{b_i\} \subset V$ is a base, $(\langle b_i, b_j \rangle)$ is the matrix of metric under this base. Then, we can define:

$$F(y) := \sqrt{\langle y, y \rangle} = \sqrt{a_{ij} y^i y^j}.$$

it can be easily verified that it is a Minkowski norm on $V$.

1.3.2. *Rander Norm.*

**Lemma 1.1.** *Suppose $\alpha$ is defined as above is Euclidean norm in Euclidean space $V$, $\beta$ is a 1-form on $V$. Then the following conditions are equivalent:*

*(1)* $F(y) = \alpha(y) + \beta(y) > 0, \quad y \neq 0,$
*(2)* $\|\beta\|_\alpha := \sqrt{a^{ij} b_i b_j} = \sqrt{a_{ij} b^i b^j} < 1,$
*(3)* $(g_{ij}(y))$ *is positive defines for all* $y \neq 0$.

*Proof.* (1) ⇒ (2) : Assume that

$$F(y) = \alpha(y) + \beta(y) = \sqrt{a_{ij} y^i y^j} + b_i y^j > 0$$

Set $y^i = -b^i$, where $b^i = a^{ij} b_j$. Then

$$F(y) = \sqrt{a_{ij} b^i b^j} - b_i b^i = \|\beta\|_\alpha - \|\beta\|_\alpha^2 = \|\beta\|_\alpha (1 - \|\beta\|_\alpha) > 0.$$

So, $\|\beta\|_\alpha < 1$.

(2) ⇒ (1) : If $\|\beta\|_\alpha < 1$, $|\beta(y)| = |b_i y^i| = |a_{ij} b^j y^i|$(*Euclidean Metric*)
$\leq |a_{ij} b^i b^j| \cdot |a_{ij} y^i y^j| = \|\beta\|_\alpha \|y\|_\alpha = \alpha(y)\|\beta\|_\alpha < \alpha(y)$. So, $F(y) = \alpha(y) + \beta(y) > 0$.

(1) $\Rightarrow$ (3) : Assume that $F(y) = \alpha(y) + \beta(y) > 0$, $y \neq 0$. Set $F_t(y) = \alpha(y) + t\beta(y)$, $0 < t < 1$. Then by Lemma 1.1.1, (See, [1]), or proposition 1.3, we have

$$F_t(y) > 0, \ \det(g_t) = (\frac{F_t}{\alpha})^{n+1} \det(a_{ij}) > 0.$$

So, every eigenvalue of $g_t \neq 0$.

Note that

- Every eigenvalue of $g_i$ is continuous,
- $g_t|_{t=0} = g$.

If there exits a eigenvalue $\lambda(t)$ of $g_t$, such that $\lambda(t)|_{t=t_1} < 0$. But as $\lambda(t)|_{t=0} > 0$, then there must be an $t_2, 0 < t_2 < t_1$, such that $\lambda(t_2) = 0$, this is a contradiction. Finally, we have every eigenvalue is positive. Particularly for $t = 1$, so $g$ is positive definite at $t = 1$.

(3) $\Rightarrow$ (1) : $F^2(y) = g_{ij}y^iy^j > 0$, for all $y \neq 0$. So $F > 0$. □

1.3.3. *$(\alpha, \beta)$-norm.*

**Theorem 1.3.** *$F = \alpha\varphi(\alpha/\beta)$ is a Minkowski norm for any Riemnnian metric $\alpha$ and 1−form $\beta$ with $\|\beta\|_\alpha < b_0$ if and only if $\varphi = \varphi(s)$ satisfies the following condition*

$$\varphi(s) > 0, \quad (\varphi(s) - s\varphi'(s)) + (b^2 - s^2)\varphi''(s) > 0,$$

*where $s$ and $b$ are arbitrary numbers with $|s| \leq b < b_0$.*

1.3.4. *A special kind of $(\alpha, \beta)$-Norm.*

$$F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}, \quad \phi(s) = 1 + \varepsilon s + ks^2.$$

1.4. **Navigation Problem.** As we already know $S_\phi := \{y\,|\phi(y) = 1\}$. Denote $S_\phi + \{v\} := \{y|\phi(y - v) = 1\}$. By okuba's method, there exists a Minkowski norm $F(y)$ satisfying

$$\phi(\frac{y}{F(y)} - v) = 1, \quad \phi(y - F(y)v) = F(y)$$

It is clearly that $F(y) = 1$ if and only if $\phi(y - v) = 1$.

Now, for any $y \in V$, we have

$$\phi(\frac{\lambda y}{F(\lambda y)} - v) = 1 \text{ and } \phi(\frac{\lambda y}{\lambda F(y)} - v) = 1.$$

So

$$\frac{\lambda y}{F(\lambda y)} - v, \frac{\lambda y}{\lambda F(y)} - v \in S_\phi,$$

Note that they have the same direction. It follows that $F(\lambda y) = \lambda F(y)$.

1.5. **Some Supplement.**

**Proposition 1.3.** *Suppose*

- *$(Q_{ij})$ is a nonsingular $n \times n$ complex matrix with inverse $(Q^{ij})$.*

- *$C = (c_j)$ be an n-vector, with $j = 1, \cdots, n$, $c_j$ are n complex numbers.*

*Let us define $C^s := Q^{sj} C_j$, then*

- $\det(Q_{ij} + CC^t) = (1 + C^s C_s)\det(Q_{ij})$.
- *Whenever $1 + C^s C_s \neq 0$, the matrix $(Q_{ij} + CC^t)$*

*is invertible. In that case, its inverse is*

$$\left(Q^{ij} - \frac{1}{1 + C^s C_s} C^i C^j\right).$$

*Remark* 1.1.

- In [5], the above proposition was stated for symmetric nonsingular matrices $(Q_{ij})$. This symmetry hypothesis on $Q$ is removable.
- $Q$ certainly need not be positive-definite.

*Proof.* The asserted formula for the inverse is correct, because multiplying that on the right by $(Q_{jk} + C_j C_k)$ indeed gives $\delta^i_k$. Thus it remains to compute the determinant of $(Q_{ij} + C_i C_j)$.

Let $C$ (resp., $C^t$) stand for the column (resp., row) vector whose entries are $C_S$, $S = 1, \cdots, n$. Then

$$(Q_{ij} + C_i C_j) = Q + CC^t.$$

Hence

$$\begin{aligned} \det\{Q + CC^t\} &= \det\left\{Q[I + (Q^{-1}C)C^t]\right\} \\ &= \det Q \det\left\{I + (Q^{-1}C)C^t\right\}. \end{aligned}$$

Now we invoke a matrix fact which is perhaps better known. It says that if $v$ and $w$ are column vectors, then

$$\det\{I + wv^t\} = 1 + v^t w. \tag{1.1}$$

Suppose neither $v$ nor $w$ is identically zero, or else there is nothing to check. Given that, there are two cases to be analyzed.

- $v^t w \neq 0$: One can find $n - 1$ linearly independent complex column vectors $w_\alpha$, such that $v^t w_\alpha = 0$. The vectors $w_\alpha$, together with $w$, form a basis for $C^n$. Consider the matrix $I + wv^t$. It has the $w_\alpha$ as eigenvectors, of eigenvalue 1. The vector $w$ is also an eigenvector, but with eigenvalue $1 + v^t w$. This observation gives (1.1).
- $v^t w = 0$: As agreed, we can suppose that both $v$ and $w$ are nonzero. Thus there is a complex column vector $u$ such that $u^t w \neq 0$. Abbreviate $v + \varepsilon u$ as $v_\varepsilon$. Note that$v^t_\varepsilon w = \varepsilon u^t w \neq 0$ for all nonzero $\varepsilon$. And in that case, we already know that $\det\{I + wv^t_\varepsilon\} = 1 + v^t_\varepsilon w$. Letting $\varepsilon \to 0$ gives (1.1).

Let us substitute, into the above, $Q^{-1}C$ for $w$ and $C$ for $v$. The said matrix fact then tells us that

$$\det\left\{I + (Q^{-1}C)C^t\right\} = 1 + C^t(Q^{-1}C) = 1 + C_sC^s,$$

which finishes the proof. □

## 2. [2]FINSLER METRIC II

**Definition 2.1.** Let $M$ be a n-dimension manifold, a *Finsler metric* on $M$ is a functional $F : TM \to [0, +\infty)$ satisfying the following properties

- F(x, y) is $C^\infty$ on $TM \setminus \{0\}$,
- For all $x \in M$, $F_x(y) \stackrel{\text{def}}{=} F(x, y)$ is a Minkowski norm.

we call $(M, F)$ is a *Finsler manifold*.

If $F(x, -y) = F(x, y)$ ie. $F(x, \lambda y) = \lambda F(x, y), \lambda \neq 0$, then we call $F$ is a *reversible finsler metric*.

### 2.1. Fundamental Tensor.

For $s \in M$, take frame $\{b_i\}_{i=1}^n$. Under this frame, $F(x, y) = F(x, y^ib_i) \stackrel{\text{def}}{=} F(x^i, y^i)$.

Let $g_{ij}(x, y) := \left(\frac{1}{2}F^2\right)_{y^iy^j}(x, y)$ is called the *fundamental tensor*.

For any $y \in T_xM$, $g_y(u, v) = g_{ij}(x, y)(u^i, u^j)$, particularly, take $u, v = y$, we have

$$F(F_{y^i}) = g_{ij}(x, y)y^j, \quad F^2(x, y) = g_{ij}(x, y)y^iy^j,$$

ie,

$$F(x, y) = \sqrt{g_{ij}(x, y)y^iy^j}.$$

*Remark* 2.1.

- Riemannian: $ds^2 = g_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$,
- Finsler: $g_y = g_{ij}(s, y)\mathrm{d}x^i \otimes \mathrm{d}x^j$.

Note that For any $y \neq 0, y \in T_xM$, $g_y$ is not uniquely depended on $y$.

*Remark* 2.2 (Special Finsler Metric).

- Riemannian metric: $F(x, y) = \sqrt{g_{ij}(x)y^iy^j}$,
- Local Minkowski metric: $F(x, y) = \sqrt{g_{ij}(y)y^iy^j}$.

*Remark* 2.3. Let $(M, F)$ is a finsler manifold. If $F$ is non-Riemannian finsler metric, then $(T_xM, F_x)$ is a Minkowski space; If $F$ is a Riemannian metric, $(T_xM, F_x)$ is a Euclidean space.

[2]Second Time of Lecture, Thursday, March 18, 2010

## 2.2. **Some Examples.**

*Example* 2.1. Riemannian metric of constant curvature. See 1.22-1.24.

*Remark* 2.4. Note about 1.24: $S^n \subset \mathbf{R^{n+1}}$. For $n = 3$, $S^3 \subset \mathbf{R^4}$. $\mathrm{d}s^2 = \sum_\alpha \mathrm{d}u^\alpha \otimes \mathrm{d}u^\alpha \overset{\mathrm{def}}{=} \sum_\alpha (\mathrm{d}u^\alpha)^2$,

$$\psi_\pm|_{S^3} : u^1 = \frac{x^1}{\sqrt{1+|x|^2}}, u^2 = \frac{x^2}{\sqrt{1+|x|^2}}, u^3 = \frac{x^3}{\sqrt{1+|x|^2}}, u^4 = \frac{\pm 1}{\sqrt{1+|x|^2}}$$

So the Riemannian metric on $S^3$ induced by $\mathrm{d}s^2$ is:

$$g = \mathrm{d}s^2|_{S^3} = \sum_\alpha (\mathrm{d}u^\alpha)^2 = \sum_{i=1}^4 (\mathrm{d}x^i)^2.$$

and, we have

$$\text{For any } y \in T_x S^3 \cong T_x R^3, F(x,y) = \sqrt{g(y,y)}.$$

The Riemannian metrics in Example 1.2.2, 1.2.3 and 1.2.4 can be expressed in one single formula.

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x,y\rangle^2)}}{1+\mu|x|^2}, \quad y \in T_x B^n(r_\mu) \cong \mathbf{R^n}$$

$$r_\mu = \begin{cases} -\frac{1}{\sqrt{-\mu}}, & \mu < 0; \\ +\infty, & \mu \geq 0. \end{cases}$$

$a_{ij}(x) = \frac{1}{1+\mu|x|^2}\left\{\delta_{ij} - \frac{\mu x^i x^j}{1+\mu|x|^2}\right\}$. By Proposition 1.3, we can get

$$a^{ij}(x) = (1+\mu|x|^2)(\delta_{ij} + \mu x^i x^j)$$

and Christoffel symbols

$$r^k_{jk} = \mu \frac{x^j \delta^i_k + x^k \delta^i_j}{1+\mu|x|^2}.$$

*Example* 2.2. $(\alpha,\beta)$-metric: Suppose $(V, |\cdot|)$ is norm space, its dual space is $(V^*, |\cdot|^*)$, $V^* := \{\text{linear function } f : V \to \mathbf{R}\}$. where $|f|^* = \sup_y \{|f(y)| : |y| = 1\} = \sup_{y\in V\setminus\{0\}} \left\{\frac{|f(y)|}{|y|}\right\}$.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)\mathrm{d}x^i$, $\beta(y) = b_i(x)y^i$ is 1-form.

Let $|\beta|_\alpha = \sup_{y\in T_xM} \frac{\beta(x,y)}{\alpha(x,y)} = \sqrt{a^{ij}(x)b_i b_j}$ (as we know the matrix of metric in $V$ and $V^*$ is reverse of each other.)

Consider $F(x,y) = \alpha\phi(\frac{\beta}{\alpha})$, $\phi = \phi(s)$ satisfy

$$\phi(s) > 0,\ \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,\ |s| \leq b < b_0.$$

Then $F(x,y)$ is a finsler metric (called $(\alpha,\beta)$ metric) if and only if $|\beta|_\alpha < b_0$.

In particular,

- If $\phi = 1 + s$, $F = \alpha + \beta$, this is a Randers metric,

- If $\phi = (1+s)^2$, $F = \frac{(\alpha+\beta)^2}{\alpha}$, it has negative curvature, and is projectively flat. ie, the geodisical is lines.
- *Berwald metric.* Let $\bar{\alpha} = \frac{\sqrt{|y|^2-(|x|^2|y^2|-\langle x,y\rangle^2)}}{1-|x|^2}$, $\bar{\beta} = \frac{\langle x,y\rangle}{1-|x|^2}$, $\lambda = \frac{1}{1-|x|^2}$, $\alpha = \lambda\bar{\alpha}$, $\beta = \lambda\bar{\beta}$. Then, the metric in (1.17) is:
$$F = \frac{(\alpha+\beta)^2}{\alpha}.$$
We should note that the metric defined above has the property:
  - $k = 0$,
  - Projectively flat,
  - Positively complete.

## 2.3. **Length Structure and Volume Form.**

2.3.1. *Length Structure.* $(M, F)$ is a finsler manifold, for any curve $C : [a,b] \to M$, the length
$$L_F(C) := \int_a^b F(x(t), x'(t))\mathrm{d}t.$$
For any $p, q \in M$, define
$$\mathrm{d}(p,q) := \inf_{C=\overline{pq}} L_F(C).$$
In general,
$$\mathrm{d}(p,q) \neq \mathrm{d}(q,p)$$
that's means that the metric is not reversible.

Conversely, given a distance functional $\mathrm{d}_F$. We can define a finsler metric:
$$F(x,y) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{d}_F(x, c(\varepsilon))}{\varepsilon}, y \in T_xM.$$
$C$ is curve with $c(0) = x, c'(0) = y$.

2.3.2. *Volume Form.* Suppose $(M, F)$: is a finsler manifold, Let local frame $\{b_i\} \in T_xM \overset{dual}{\leftrightarrow} \left\{\theta^i\right\} \in T_xM^*$.

- Define
$$\mathrm{d}V_{BH} := \delta_{BH}(x)\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n.$$
where,
$$\delta_{BH}(x) := \frac{\mathrm{vol}(B^n(1))}{\mathrm{vol}\left\{(y^i) \in \mathbf{R}^n | F(x, y^i b_i) < 1\right\}}$$
which is called Busemamn-Hausdorff volume form.
- Define
$$\mathrm{d}V_{HT} := \delta_{HT}(x)\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n$$
where
$$\delta_{HT}(x) := \frac{B_x^n \det(g_{ij}(x,y))\mathrm{d}y^1 \cdots \mathrm{d}y^n}{\mathrm{vol}(B^n(1))}$$

where $B^n_x := \left\{(y^i)\in \mathbf{R}^n | F(x, y^i b_i) < 1\right\}$.

we called it is the Holmes-Thompson volume form.

In general, The coefficient of volume form can't be expressed in terms of elementary functions though $F = F(x, y)$ sometimes is. Nevertheless, it is computable for Randers metrics including Riemannian metrics.

First, consider a Riemannian metric

$$\alpha = \sqrt{a_{ij}(x)y^i y^i}, \quad y = y^i \frac{\partial}{\partial x^i}\bigg|_x \in T_x M.$$

there $(a_{ij}(x)$ is positive definite, ie. there is an invertible matrix $A \in M_n(\mathbf{R})$ such that $(a_{ij}(x) = A^T A))$.

Take transformation

$$\mathbf{R^n} \to \mathbf{R^n}$$

$$y \mapsto u = Ay$$

then, for any $y \in B^n_x = \left\{(y^i)\in \mathbf{R^n} | \alpha(\mathbf{x}, \mathbf{y^i b_i}) < \mathbf{1}\right\}$, $|u|^2 = u^T u = y^T A^T A y = y^T(a_{ij}(x))y = a_{ij}(x)y^i y^j < 1$. that's mean $A\colon B^x_n \to B^n(1)$.

Then

$$\begin{aligned} \mathrm{vol}(B^n(1)) \int_{B^n(1)} \mathrm{d}u^1 \cdots \mathrm{d}u^n &= \int_{B^n(x)} \det(A)\mathrm{d}y^1 \cdots \mathrm{d}y^n \\ &= \sqrt{\det(a_{ij})}\mathrm{vol}(B^n_x). \end{aligned}$$

That's

$$\sigma_{BT}(x) = \frac{\mathrm{vol}(B_n(1))}{\mathrm{vol}(B^n_x)} = \sqrt{\det(a_i j)}$$

$$\mathrm{d}v_\alpha = \sqrt{\det(a_{ij})}\theta^1 \wedge \cdots \wedge \theta^n$$

If $F$ is a Randers metric, set $F = \alpha + \beta$, $\alpha = \sqrt{\det(a_{ij})}$, $\beta = b_i y^i$. Taking orth-normal frame $\{e_i\} \subset T_x M$, w.r.t. $\alpha$ such that $\beta(y) = |\beta|_\alpha y^1$.

In fact, taking $\{e_i\} \overset{\text{dual}}{\longleftrightarrow} \left\{w^i\right\}$, where

$$w^i = \frac{b_i(x)}{|\beta|_\alpha}\mathrm{d}x^i, \text{ ie, } e_i = \frac{b_i(x)}{|\beta|_\alpha}\frac{\partial}{\partial x^i}, \; b^i := a^{ij}b_j, y = y^i e_i.$$

Then

$$1 > F(x,y) = \sqrt{\sum_i (y^i)^2} + |\beta|_\alpha y^1$$

$$\sum_i (y^i)^2 < 1 + |\beta|_\alpha^2 (y^1)^2 - 2|\beta|_\alpha y^1$$

$$(1-|\beta|_\alpha^2)^2 (y^1 + \frac{|\beta|_\alpha}{1-|\beta|_\alpha^2})^2 + (1-|\beta|_\alpha^2)\sum_{\alpha=2}^n (y^\alpha)^2 < 1$$

Now we can take an properly transformation to get $\cdots$

In fact, let

$$\begin{cases} u^1 = (1-|\beta|_\alpha^2)(y^1 + \frac{|\beta|_\alpha}{1-|\beta|_\alpha^2}), \\ u^\alpha = \sqrt{1-|\beta|_\alpha^2} y^\alpha, \qquad \alpha \geq 2. \end{cases}$$

The determinant of its Jacob matrix is $(1-|\beta|_\alpha)^{(n+1)/2}$.

So, we have

$$\begin{aligned} \mathrm{vol}(B^n(1)) &= \int_{B^n(1)} \mathrm{d}u^1 \cdots \mathrm{d}u^n \\ &= \int_{B_x^n} (1-|\beta|_\alpha)^{(n+1)/2} \mathrm{d}y^1 \cdots \mathrm{d}y^n \\ &= (1-|\beta|_\alpha)^{(n+1)/2} \mathrm{vol}(B_x^n). \end{aligned}$$

That's

$$\sigma_{BT}(x) = \frac{\mathrm{vol}(B_n(1))}{\mathrm{vol}(B_x^n)} = (1-|\beta|_\alpha)^{(n+1)/2}.$$

Note that there is a little different between the text book and here, as we take the coordinate as the orth-normal, such that $\sigma_\alpha(x) = 1$.

## 3. [3]Navigation Problem

(1926, zermelo) On a Euclidean space $(M, \Phi)$, with a fix direction $u$, and a velocity $\Phi(u) = 1$, then form $A$ to $B$ the curve with minimal distance (length) is line. Now suppose that there is a external force (wind), and set the direction fixed, with speed $v$, what about the minimum distance should be?

Set $(M, \Phi)$ is a finsler manifold, and velocity vector field $u$, with $\Phi(u) = 1$. Suppose there is a external force field $v$, such that $\Phi(x, -v_x) < 1$. As $u$ is a unit vector, we have $\Phi(x, T_x - v_x) = \Phi(x, u_x) = 1$, note that $\{x|\Phi(x, u_x) = 1\}$ is the indictrix of $\Phi$. In order to make $T_x$ to be a unit vector, the equation $\bar{\Phi}(x, T_x) = \Phi(x, T_x - v_x) = 1$ should be the indictrix of the new metric. In fact it is a shifting of $\{x|\Phi(x, u_x) = 1\}$.

[3]Third Time of Lecture, Thursday, March 25, 2010

By Okuba's method, there exist a finsler metric $F$ such that $\Phi(x, y/F(x,y) - v_x) = 1$, equally $F(x,y) = \Phi(x, y - F(x,y)v_x)$. It's easy to see that $F(x,y) = 1$ if and only if $\Phi(x, y - v_x) = 1$ and $F(x, T_x) = 1$.

**Lemma 3.1** (lemma 1.4.2, [1]). *Let $(M, \Phi)$ be a finsler manifold, a vector field $v$ with $\Phi(x, -v_x) < 1$. Define a finsler metric $F$ satisfying $\Phi(x, y/F - v_x) = 1$. Then, we have*

$$\mathrm{d}v_F = \mathrm{d}v_\Phi \quad (\textit{Busemanmn-Hausdorff volume form}),$$

*where*

$$\sigma_{BH}(x) := \frac{\mathrm{vol}(B^n(1))}{\mathrm{vol}(\{(y^i) \in \mathbf{R}^n | F(x, y^i\alpha_i) < 1\})}.$$

Recall $(V, |\cdot|) \leftrightarrow (V^*, |\cdot|^*)$, and for any $f \in V^*$, $\|f\|^* := \sup_{y \in V\setminus\{0\}} f(y)/\|y\|$.

*Remark* 3.1. Suppose $(M, F)$ is a finsler manifold, and $v$ is a vector field on $M$ with $\Phi(x, v_x) < 1$. Define

$$\Phi^*(x, \xi) = \sup_{y \in T_xM\setminus\{0\}} \frac{\xi(y)}{\Phi(x,y)}$$

we can view $\Phi \simeq \Phi^*$ is the dual metric of $\Phi^*$.

$v\colon u^* \mapsto \langle v, u^* \rangle \overset{\text{def}}{=} u^*(v)$, the tangent vector is a liner function of cotangent space. Define

$$\begin{array}{ccccc} F^* & := & \Phi^* & + & v^* \\ \updownarrow & & \downarrow & & \downarrow \\ F & \overset{\text{def}}{\leftarrow} & (\Phi & , & v) \end{array}$$

where $F$ defined by $\Phi(x, y/F - v_x) = 1$. This mean that we can use $\Phi \to \Phi^*$, $v \to v^*$ to define $F^* \overset{\text{def}}{=} \Phi^* + v^*$, and then define $F$ by $F^*$. See MSR1 series 2004 Z.Shen *The Property of Rich Curvature and Flag Curvature of Finsler Manifolds*.

3.1. **Navigation Problem in Two Special Kinds of Spaces.** Now we turn to the navigation problem on two special kinds spaces, which will induce two special finsler metric.

3.1.1. *Frunk metric.* Let $\phi = \phi(y)$ be a Minkowski norm in $\mathbf{R^n}$. Define domain

$$\mathscr{U} := \left\{(y^i) \in \mathbf{R^n} | \phi(y) < 1\right\}$$

in fact it's the domain defined by the boundary of the indictrix of Minkowski norm, and so it's a *strong convex domain*. Taking $\Phi = \Phi(x,y) := \phi(y)$, and vector field $v$ in $\mathbf{R^n}$ such that $v_x = -x$. By navigation problem, there is a $\Theta$, such that $\Phi(x, y/\Theta - v_x) = 1$, i.e. $\phi(y/\Theta(x,y) + x) = 1$. So

$$\Theta(x,y) = \phi(y + \Theta(x,y)x), \tag{3.1}$$

differentiating (3.1) with respect to $x^k$ and $y^k$ respectively, one obtains

$$(1-\phi^l_w(w)x^l)\Theta_{x^k}(x,y)=\phi_{w^k}(w)\Theta(x,y), \tag{3.2}$$

$$(1-\phi^l_w(w)x^l)\Theta_{y^k}(x,y)=\phi_{w^k}(w), \tag{3.3}$$

where $w:=y+\Theta(x,y)x$. By (3.2) and (3.3), $\phi_{w^k}$ and $1-\phi_{w^l}x^l$ are not zero at the same time. See $1-\phi_{w^l}x^l$ and $\phi_{w^l}$ be unknown number, then

$$\begin{vmatrix} \Theta_{x^k} & -\Theta \\ \Theta_{y^k} & -1 \end{vmatrix}=0,$$

from which one obtain

$$\Theta_{x^k}=\Theta\Theta_{y^k}.$$

Which called the basic equation(differential) of Frunk metric. Particularly, when $\phi$ is the standard Euclidean metric, then $\mathscr{U}$ is the unit open ball in usual sense and $\Theta$ is the Frunk metric defined in (1.15) (See [1]).

*Remark* 3.2. In fact, in the Euclidean case, $\phi(y)=\sum(y^i)^2$, and $\Theta$ is defined by $x+y/\Theta(x,y)\in\partial\mathscr{U}$. Thus, $\phi(x+y/\Theta)=1$, ie., $|x|^2+2/\Theta\langle x,y\rangle+|y|^2/\Theta^2=0$, from this, we can get

$$\Theta(x,y)=\frac{\langle x,y\rangle+\sqrt{\langle x,y\rangle^2+(1-|x|^2)|y|^2}}{1-|x|^2}.$$

which is exactly the metric defined in (1.15).

3.1.2. Randers metric. . Give a Riemannian manifold $(M,h)$, $h=\sqrt{h_{ij}(x)y^iy^j}$ and a vector field $V$ with $h(x,-V_x)<1$. Solve

$$h(x,y/F-V_x)=1\Leftrightarrow\sqrt{h_{ij}(y^i/F-V^i)(y^j/F-V^j)}=1,$$

The solution[4] of navigation problem $F = \alpha + \beta$, where

$$\alpha = \sqrt{a_{ij}(x)y^i y^j}, \quad a_{ij} = \frac{(1 - h_{pq}V^p V^q)h_{ij} + h_{ip}h_{jq}V^p V^q}{(1 - h_{pq}V^p V^q)^2},$$
$$\beta = b_i(x)y^i, \quad b_i = \frac{h_{ip}V^p}{1 - h_{pq}V^p V^q}.$$

write $\lambda := 1 - \|V\|_h^2$, $V_i = h_{ij}V^j$, $V_0 := V_i y^i$, $h = \sqrt{h_{ij}y^i y^j}$.

$$F(x, y) = \frac{\sqrt{h^2 - (\|V\|_h^2 h^2 - V_0^2)} - V_0}{\lambda} = \frac{\sqrt{\lambda h^2 + V_0^2}}{\lambda} - \frac{V_0}{\lambda},$$

note that, $\frac{\sqrt{\lambda h^2 + V_0^2}}{\lambda}$ and $\frac{-V_0}{\lambda}$ is the basic tensors of $a_{ij}$ and $b_i$ respectively.

Then,

$$\begin{cases} a_{ij} &= \frac{h_{ij}}{\lambda} + \frac{V_i}{\lambda}\frac{V_j}{\lambda}, \quad (\text{use } \alpha(x, y) = \sqrt{a_{ij}y^i y^j}) \\ b_i &= -\frac{V_i}{\lambda}, V_i = h_{ij}V^j. \end{cases}$$

By Proposition 1.3,

$$\begin{cases} a^{ij} = \lambda(h^{ij} - V^i V^j), \\ b^i = a^{ij}b_i = -\lambda V^i, \end{cases}$$

[4]In fact, by

$$F^2 \left(h_{ij}V^i V^j\right) - h_{ij}\left(y^i V^j + y^j V^i\right)F + h_{ij}y^i y^j = F^2,$$
$$F = \frac{h_{ij}y^i V^j \pm \sqrt{\left(h_{ij}y^i V^j\right)^2 + \left(1 - h_{pq}V^p V^q\right)h_{ij}y^i y^j}}{h_{ij}y^i y^j - 1},$$

and note

$$h_{pq}V^p V^q = h(x, -V_x)^2 < 1,$$
$$F = \frac{-h_{ij}y^i V^j + \sqrt{h_{ij}y^i V^j h_{pq}y^p V^q - h_{ij}y^i y^j h_{pq}V^p V^q + h_{ij}y^i y^j}}{1 - h_{ij}y^i y^j},$$

thus,

$$F = \frac{-h_{ij}y^i V^j + \sqrt{h_{ip}y^i V^p h_{jq}y^j V^q - h_{ij}y^i y^j h_{pq}V^p V^q + h_{ij}y^i y^j}}{1 - h_{ij}y^i y^j},$$

ie.,

$$F = \frac{-h_{ij}y^i V^j + \sqrt{\left(h_{ip}h_{jq}V^p V^q + \left(1 - h_{pq}V^p V^q\right)h_{ij}\right)y^i y^j}}{1 - h_{ij}y^i y^j},$$

we have

$$F = \sqrt{\frac{\left(1 - h_{pq}V^p V^q\right)h_{ij} + h_{ip}h_{jq}V^p V^q}{\left(1 - h_{pq}V^p V^q\right)^2}y^i y^j} + \frac{-h_{ip}V^p}{1 - h_{pq}V^p V^q}y^i.$$

and

$$\|\beta\|_\alpha^2 = a^{ij}b_ib_j = h_{ij}V^iV^j = h^2(x, v_x) < 1.$$

so we can feel that Randers metric is a special, natural (the solution of navigation problem) metric. We get the following

**Proposition 3.1.** *The solution of navigation problem on a Riemannian manifold is a Randers metric, and conversely, every Randers metric can induced by a navigation problem on a Riemannian manifold.*

*Proof.* One side already given as above, we just need to do the other side. Suppose $F = \alpha + \beta$ is a Randers metric. Where $\alpha = \sqrt{a_{ij}(x)y^iy^j}, \beta = b_i(x)y^i$. Put,

$$h_{ij}(x) := (1 - \|\beta\|_\alpha^2)\left(a_{ij} + b_ib_j\right),$$

$$V^i(x) := -\frac{a^{ij}b_j}{1 - \|\beta\|_\alpha^2},$$

form them we can induce the navigation data $(h, v) \to F = \alpha + \beta$, you should verify yourself.

In fact, as

$$a_{ij} = \frac{\left(1 - h_{pq}V^pV^q\right)h_{ij} + h_{ip}h_{jq}V^pV^q}{\left(1 - h_{pq}V^pV^q\right)^2},$$

$$b_i = -\frac{h_{ip}V^p}{1 - h_{pq}V^pV^q}.$$

It's easy to get

$$\begin{aligned} h_{ij} &= \left(1 - h_{pq}V^pV^q\right)\left\{a_{ij} - b_ib_j\right\}, \\ &= \left(1 - \|\beta_x\|_\alpha^2\right)\left\{a_{ij} - b_ib_j\right\}, \end{aligned}$$

and

$$\begin{aligned} a^{ij}b_j &= a_{ij}\left(-\frac{h_{jp}V^p}{1 - \|\beta_x\|_\alpha^2}\right) \\ &= -a^{ij}\left(a_{jp} - b_jb_p\right)V^p \\ &= -V^i + a^{ij}b_jb_pV^p, \end{aligned}$$

note

$$b_pV^p = -\frac{h_{pj}V^jV^p}{1 - \|\beta_x\|_\alpha^2},$$

we have

$$V^i = \frac{a^{ij} b_j}{1 - \|\beta_x\|_\alpha^2}.$$

□

## 4. CARTAN TORSION

4.1. **Cartan Torsion.** Suppose $(M, F)$ is a finsler manifold, for any $u, v, w \in T_x M$, define

$$C_y(u, v, w) := \frac{\partial^3}{4\partial s \partial t \partial r} \left[F^2(y + su + tv + rw)\right]_{s=t=r=0}, \quad y \in T_x M \setminus \{0\}.$$

We call

$$C := \left\{C_y | y \in V \setminus \{0\}\right\}$$

to be *Cartan tensor*.

In local coordinate, $C$ can be expressed exactly. Taking frame $\{\mathbf{b_i}\} \in T_x M$. Then,

$$g_{ij}(x, y) = g_y(\mathbf{b_i}, \mathbf{b_j}) = \frac{\left[F^2\right]_{y^i y^j}}{2} = \frac{1}{2} \frac{\partial \left[F^2\right]}{\partial y^i \partial y^j},$$

$$C_{ijk}(x, y) = C_y(\mathbf{b_i}, \mathbf{b_j}, \mathbf{b_k}) = \frac{1}{4} \frac{\partial^3 \left[F^2\right]}{\partial y^i \partial y^j \partial y^k} = \frac{\partial g_{ij}}{2\partial y^k}.$$

*Remark* 4.1. $C = 0 \Leftrightarrow F$ is a Euclidean metric, i.e. independent of $y$.

*Remark* 4.2. $T_x M$ have a smooth structure that make it into a manifold. For any $y \in T_x M$, we know that $T_y(T_x M) \cong T\mathbf{R}^\mathbf{n}$. Define

$$g_{ij} := g_{ij}(x, y) \mathrm{d}y^i \otimes \mathrm{d}y^j,$$

which is the inner product defined on tangent space of $T_x M$ at $y$. Particularly, if $(g_{ij})$ is a Riemannian metric on $T_x M$, then $(T_x M, g_x)$ is a Riemannian manifold, and the coefficients of this Riemannian manifold is the Christoffel symbol.

In general, the Christoffel symbol on $(T_x M, g_x)$ is

$$C^i_{jk} = g^{il} C_{ljk}$$

which means that the Cartan tensor determined the Christoffel symbol on its tangent space (as a Riemannian manifold).

4.2. **Mean Cartan Torsion.** For any $y \in T_x M \setminus \{0\}$,

$$\begin{aligned} \mathbf{I}_y : T_x M &\to \mathbf{R} \\ u &\mapsto \mathbf{I}_y(u) := g^{jk} C_y(u, \mathbf{b_j}, \mathbf{b_k}) \end{aligned}$$

$\mathbf{I}_y$ is a liner function on tangent space, put $\mathbf{I}_y = T_i(x,y)\mathrm{d}\theta^i$, there $\left\{\theta^i\right\}$ is the basis of co-tangent space. Observe

$$\begin{aligned}\frac{\partial}{\partial y^i}\left[\det(g_{jk})\right] &= \frac{\partial g_{pq}}{\partial y^i}A_{pq} = \frac{\partial g_{pq}}{\partial y^i}\left[g^{q_1p}g_{pq_1}\right]A_{pq}\\ &= \frac{\partial g_{pq}}{\partial y^i}g^{q_1p}\delta^p_{q_1}\det(g_{jk}) = \frac{\partial g_{pq}}{\partial y^i}g^{qp}\det(g_{jk})\\ &= 2\det(g_{jk})g^{pq}C_{pqi},\end{aligned}$$

we have

$$I_i(x,y) = g^{jk}C_{ijk} = \frac{\partial}{\partial y^i}\left[\ln\sqrt{\det(g_{ij})}\right].$$

In fact, Deicke have proved that Euclidean norms can be characterized by the mean Cartan torsion, proof can be find in [2].

*Remark* 4.3 (*Oek theorem*). A Minkowski norm on a vector space $V$ is Euclidean if and only if $I = 0$.

4.3. ***Matsumoto tensor.*** Define

$$\begin{aligned}\mathcal{M}_{ijk} &:= C_{ijk} - \frac{1}{n+1}\left(I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\right),\\ h_{ij}(x,y) &:= g_{ij}(x,y) - l_il_j,\\ l_i &= \frac{y_i}{F(x,y)}, y_i = g_{ij}(x,y)y^j,\end{aligned}$$

Note that

$$g_{ij} = \frac{1}{2}\left[F^2\right]_{y^iy^j},$$

we have

$$y^ig_{ij} = \frac{1}{2}\left[F^2\right]_{y^iy^j}y^i = \frac{1}{2}\left[F^2\right]_{y^j} = FF_{y^j}.$$

So, $l_i = F_{y^i}$ is the unit vector determined by $y$, and $h_{ij}(x,y) = g_{ij} - F_{y^i}F_{y^j}$. We call $\mathcal{M}_y(u,v,w) := \mathcal{M}_{ijk}(x,y)u^iv^jw^k$ is the Matsumoto torsion. It's clearly $\mathcal{M} := \left\{\mathcal{M}_y | y \in V\setminus\{0\}\right\} \equiv 0$ for all two-dimensional Minkowski norms.(?)

*Example* 4.1 (1.5.2, Randers metric). Let $F = \alpha + \beta$, with $\|\beta_\alpha\| < 1$, and the metric matrix is $(g_{ij})$, and determinant of $(g_{ij})$ is $\det(g_{ij})$. then

$$I_i = \frac{\partial\left[\ln\sqrt{\det(g_{ij})}\right]}{\partial y^i}.$$

In fact, $F = \alpha + \beta$, $\alpha = \sqrt{a_{jk}(x)y^j y^k}$,

$$
\begin{aligned}
F_{y^i} &= \alpha_{y^i} + \beta_{y^i} = \frac{2\alpha_{jk}(x)y^j\delta^i_k}{2\alpha} + b_i \\
&= \frac{a_{ij}y^j}{\alpha} + b_i = \frac{y_i}{\alpha} + b_i, \\
I_i &= \frac{\partial}{\partial y^i} \ln \sqrt{\left(\frac{\alpha+\beta}{\alpha}\right)^{n+1} \det(a_{ij})} \\
&= \frac{1}{\sqrt{\left(\frac{\alpha+\beta}{\alpha}\right)^{n+1} \det(a_{ij})}} \frac{1}{2\sqrt{\left(\frac{\alpha+\beta}{\alpha}\right)^{n+1} \det(a_{ij})}} (n+1)\left(\frac{\alpha+\beta}{\alpha}\right)^n \left(\frac{\alpha+\beta}{\alpha}\right)_{y^i} \det(a_{ij}(x)) \\
&= \frac{1}{2\left(\left(\frac{\alpha+\beta}{\alpha}\right)\det(a_{ij})\right)}(n+1)\frac{F_{y^i}\alpha - F\alpha_{y^i}}{\alpha^2}\det(a_{ij}) \\
&= \frac{(n+1)}{2(\alpha+\beta)}\left(b_i - \frac{\beta}{\alpha^2}y_i\right),
\end{aligned}
$$

note

$$
\begin{aligned}
g_{ij} &= \frac{F}{\alpha}\left\{a_{ij} - \frac{y_i y_j}{\alpha^2} + \frac{\alpha}{F}\left(b_i + \frac{y_i}{\alpha}\right)\left(b_j + \frac{y_j}{\alpha}\right)\right\}, \\
\left[\frac{F}{\alpha}\right]_{y^k} &= \frac{\alpha b_k - y_k/\alpha \cdot \beta}{\alpha^2}, \quad \left[\frac{y_i}{\alpha}\right]_{y^k} = \frac{1}{\alpha}\left(\delta^i_k + \frac{y_i}{\alpha}\frac{y_k}{\alpha}\right),
\end{aligned}
$$

then (set $\frac{y_i}{\alpha} \overset{\text{def}}{=} a_i$, then $\left[\frac{F}{\alpha}\right]_{y^k} = \frac{1}{\alpha^2}(\alpha b_k - \beta a_k)$, $(a_i)_{y^k} = \frac{1}{\alpha}\left(\delta^i_k + a_i a_k\right)$)

$$
\begin{aligned}
C_{ijk} &= \frac{\partial g_{ij}}{\partial y^k} = \left[\frac{F}{\alpha}\right]_{y^k}\left(a_{ij} - a_i a_j\right) + \frac{F}{\alpha}\left[a_{ij} - a_i a_j\right]_{y^k} + \left[(b_i + a_i)\left(b_j + a_j\right)\right]_{y^k} \\
&= \frac{1}{\alpha}\left(b_k - \frac{\beta}{\alpha}a_k\right)\left(a_{ij} - a_i a_j\right) + \frac{F}{\alpha}\left[-(a_i)_{y^k}a_j - a_i(a_j)_{y^k}\right] + (a_i)_{y^k}\left(b_j + a_j\right) + (b_i + a_i)(a_j)_{y^k} \\
&= \frac{1}{\alpha}\left(b_k - \frac{\beta}{\alpha}a_k\right)\left(a_{ij} - a_i a_j\right) - \left(1 + \frac{\beta}{\alpha}\right)\left[(a_i)_{y^k}a_j + a_i(a_j)_{y^k}\right] + (a_i)_{y^k}\left(b_j + a_j\right) + (b_i + a_i)(a_j)_{y^k} \\
&= \frac{1}{\alpha}\left(b_k - \frac{\beta}{\alpha}a_k\right)\left(a_{ij} - a_i a_j\right) + (a_i)_{y^k}\left(b_j - \frac{\beta}{\alpha}a_j\right) + (a_j)_{y^k}\left(b_i - \frac{\beta}{\alpha}a_i\right) \\
&= \cdots = \frac{1}{n+1}\left\{I_i h_{jk} + I_k h_{ij} + I_j h_{ki}\right\},
\end{aligned}
$$

where

$$
h_{ij} = FF_{y^i y^j} = \frac{\alpha+\beta}{\alpha}\left(a_{ij} - \frac{y_i y_j}{\alpha^2})\right).
$$

Which implies that $\mathcal{M}_{ijk} = 0$. So we have the

*Proposition* 4.1. *The Matsumoto tensor of Every Randers metric is equal to 0.*

What's more, partial of the converse of the proposition is also true, i.e.

*Proposition* 4.2. *If $n \geq 3$ and the Matsumoto tensor of a metric is equal to 0, then it must be a Randers metric.*

*Remark* 4.4. *History*: In 1972, $\mathcal{M} = 0 \Rightarrow F$ is $(\alpha,\beta)$ metric. In 1978, $n \geq 3, \mathcal{M} = 0 \Rightarrow F$ is Randers metric, $\cdots$

4.4. **The Norms of $C$, $\mathcal{I}$, and $\mathcal{M}$.** We know that all of the tensors $C$, $\mathcal{I}$, $\mathcal{M}$ is the non-Riemannian geometric quantity. Define

$$\|\mathcal{I}\| = \sup_{x\in M} \|I\|_x,$$

$\|I\|_x$ defined on the $T_xM$ and is the Minkowski norm. Note that these definitions are (0)-homogenous.

**Lemma 4.1** (1.5.4, estimate of mean Cartan tensor)**.** *By definition,*

$$\|\mathcal{I}\| = \sup_{y,u\in V\setminus\{0\}} \frac{F(y)|I_y(u)|}{\sqrt{g_y(u,u)}} \xlongequal{(0)-homogenous} \sqrt{g^{ij}(y)I_iI_j}$$

$$\|\beta\|_\alpha = \sup_{y\in V\setminus\{0\}} \frac{\beta(y)}{\alpha(y)} \Rightarrow \frac{\beta(y)}{\alpha(y)} \leq \|\beta\|_\alpha$$

*So, we get*

$$\beta(y) = \|\beta\|_\alpha \alpha(y)\cos\theta \cdots \Rightarrow I_iI_j = \left(\frac{n+1}{2}\right)^2 \frac{\|\beta\|_\alpha^2 \sin^2\theta}{1+\|\beta\|_\alpha\cos\theta},$$

*Let*

$$f(\theta) = \frac{\sin^2\theta}{1+\|\beta\|_\alpha\cos\theta} = \frac{1-t^2}{1+\|\beta\|_\alpha t}, \quad t\in[-1,1], \quad f'(\theta) = \sin\theta(\cos\theta - \frac{-1\pm\sqrt{1-\|\beta\|_\alpha^2}}{\|\beta\|_\alpha})$$

*form which we can induce $\|\mathcal{I}\|^2$ attains its maximum at $\theta_0$, $f'(\theta_0) = 0$.*

## **Part** 2. **Note of Chapter 2, Structure Equation**

Connection is a tool to depict the direction derivative.

## 5. CONNECTION

5.1. **Definition of Connection.** Roughly speak, connection when restriction to a differential manifold is the direction derivative of a vector bundle.

**Definition 5.1.** Suppose $\mathcal{V}$ is a vector bundle on $N$, denote $C^\infty(\mathcal{V})$ be the set of (smooth) section on the vector bundle. The connection

$$\begin{aligned} \triangledown : C^\infty(\mathcal{V}) &\to T^*N\otimes\mathcal{V}_x \\ \mathbf{X} &\mapsto \triangledown\mathbf{X} \end{aligned}$$

satisfying

- For any $\mathbf{X}_1, \mathbf{X}_2 \in C^\infty(\mathcal{V})$, $\triangledown(\mathbf{X}_1+\mathbf{X}_2) = \triangledown\mathbf{X}_1 + \triangledown\mathbf{X}_2$,

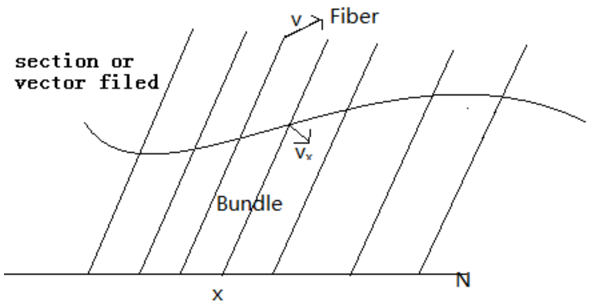


FIGURE 1. The Section of Vector Bundle

- For any $f \in C^\infty(N)$, $\mathbf{X} \in C^\infty(\mathcal{V})$

$$\nabla(f\mathbf{X}) = \mathrm{d}f \otimes \mathbf{X} + f \nabla \mathbf{X}$$

then, $\nabla$ is called the *connection* on vector bundle $\mathcal{V}$.

Let $\langle , \rangle$ is the paring between $TN$ and $T^*N$. i.e.,

$$\langle v, w \rangle = w(v), \quad v \in T_xN,\ w \in T_x^*N.$$

For any $v \in C^\infty(TN)$, $\mathbf{X} \in C^\infty(\mathcal{V})$, write

$$\nabla_v \colon C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V})$$
$$\mathbf{X} \mapsto \nabla_v\mathbf{X} := \langle v, \nabla\mathbf{X} \rangle \in C^\infty(\mathcal{V})$$

Take local basis $\{e_i\} \in C^\infty(\mathcal{V})$, that's $\{e_i(x)\}$ is a basis of fiber $\mathcal{V}_x$ for every $x$ in some neighborhood $U$ of $x$. Suppose that $x^\alpha$ is the local coordinate system, then the basis for $C^\infty(T^*N \otimes \mathcal{V})$ is $\{\mathrm{d}x^\alpha \otimes e_i\}$. Thus, we can suppose

$$\nabla e_i = \Gamma^j_{i\alpha} \mathrm{d}x^\alpha \otimes e_j,$$

write,

$$\omega_i^j := \Gamma^j_{i\alpha} \mathrm{d}x^\alpha,$$

then

$$\nabla e_i = \omega_i^j \otimes e_j,$$

in which $\omega_i^j$ is called *connection form of* $\nabla$, and $\Gamma^j_{i\alpha}$ is the *connection coefficients* of connection $\nabla$.

This is the general definition of connection, we have to constrain the general connection to some special cases, such as chern connection, Cartan connection and so on, this can be give by require the connection satisfy some condition, usually they are torsion condition and compatibility with the give metric. It has been proved that there isn't exist a connection on finsler manifold in general which is torsion free and compatible with the metric.

5.2. **Chern Connection.** Suppose $(M, F)$ is a finsler manifold. $TM = \cup_{x\in M} T_xM$ is the tangent bundle, then

$$\begin{array}{ccc}
TM_0 = TM \setminus \{0\} & \longleftarrow & T_y(TM_0) \ni (x,y,v), \\
(x,y) & & y \overset{\text{def}}{=} (x,u) \\
\downarrow \pi & & \uparrow \pi^* \\
x \in M & \longleftarrow & T_xM \ni (x,v) \\
(x,u) \in TM_0 & \longleftarrow & T(TM_0) \ni (x,y,v) \\
\pi \downarrow & & \uparrow \pi^* \\
x \in M & \longleftarrow & TM \ni (x,v)
\end{array}$$

where $T(TM_0) \overset{\text{def}}{=} \pi^*(TM) \oplus VTM$, $VTM := \text{span}\left\{\partial/\partial y^i\right\}$. $\pi^*TM\,|_{(x,y)} = \{(x,y,v) | v \in T_xM\} \cong$

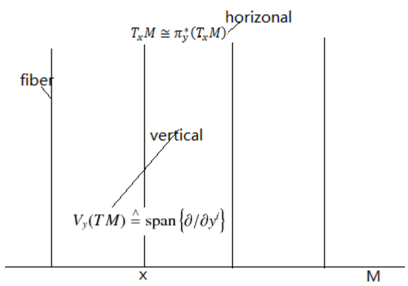


FIGURE 2. The Horizonal Part and the Vertical Part of $M$

$T_xM$,

$$\begin{array}{ccc}
\text{basis}\left\{\partial/\partial x^i\right\} \subset T_xM & \longleftrightarrow & \pi^*(TM) \supset \text{basis}\left\{\partial_i = (x,y,\partial/\partial x^i)\right\} \\
\text{dual} \updownarrow & & \text{dual} \updownarrow \\
\text{basis}\left\{\mathrm{d}x^i\right\} \subset T_x^*M & \longleftrightarrow & \pi^*(T^*M) \supset \text{basis}\left\{\mathrm{d}x^i = \left(x,y,\mathrm{d}x^i\,|_x\right)\right\}
\end{array}$$

For any $y \in TM \setminus \{0\}$, let $g_{y^i} := g_{ij}(x,y)\mathrm{d}x^i \otimes \mathrm{d}x^j$ on $T_xM$, it is a family of inner products on $TM_0$. $g := g_{ij}(x,y)\mathrm{d}x^i \otimes \mathrm{d}x^j$ is a unique inner product on $\pi^*(TM)$.

$$T(TM_0) = \pi^*(TM) \oplus VTM.$$

where $\pi^*(TM)$ is a sub-tangent bundle (vector bundle) on the tangent bundle of $M$.

*Remark* 5.1. As $g_{ij}$ depend on $y$ (if not, it must be a Riemannian metric), so it should be discussed on the tangent bundle.

**Theorem 5.1** (2.1.1, *Chern connection exist theorem*)**.** *Suppose $(M, F)$ is a finsler manifold, $\{e_i\}$ is the basis for $\pi^*(TM)$, the dual basis of $\{e_i\}$ is $\left\{\omega^i\right\} \subset \pi^*(T^*M)$. Then there is a unique*

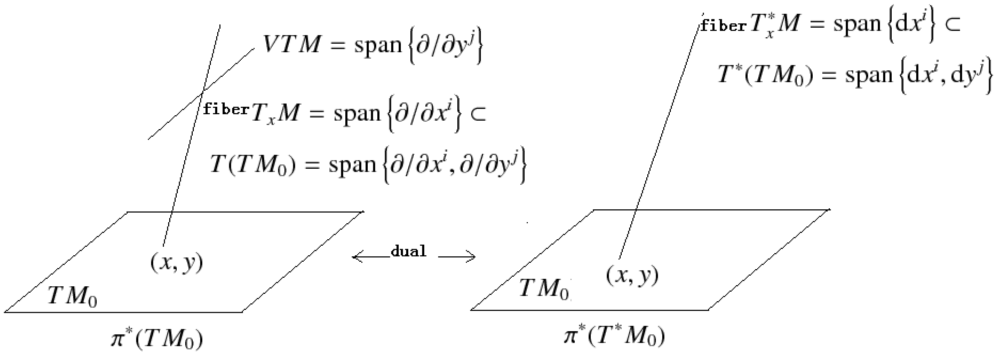


FIGURE 3. dual vector bundle and their fiber

*connection form* $\{\omega^i_j\}$ *satisfying the following condition*

(2.3) $$\mathrm{d}\omega^i = \omega^i \wedge \omega^i_j \text{—}\textit{torsion free},$$

(2.4) $$\mathrm{d}g_{ij} = g_{ik}\omega^k_j + g_{kj}\omega^k_i + 2C_{ijk}\omega^{n+k}$$

*—almost metric compatible,*

*where* $\omega^{n+k} := \mathrm{d}y^k + y^j\omega^k_j$.

*Remark* 5.2. This theorem first proofed by S.Chern in 1944, in 1948 an improvement has been given and in the same article, S.Chern complete the classification of Finsler metric. In 1958, Rund defined a new connection named *Rund connection*, which can be proofed is the same as Chern connection.

*Remark* 5.3. Define

$$\begin{aligned}(\nabla g)_{ij} &= (\nabla g(e_i, e_j)\\ &= \nabla g_{ij} - g(\nabla e_i, e_j) - g(e_i, \nabla e_j)\\ &= \mathrm{d}g_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j \quad \text{by Chern connection}\\ &= 2C_{ijk}\omega^{n+k} \quad \text{(if theorem holds).}\end{aligned}$$

thus, if a connection is torsion free, and compatible with the metric, then it must be a Riemannian metric.

Note, *Cartan connection* is compatible with the metric but it's not torsion free.

*Proof.* Taking

$$e_i := \partial/\partial x^i \overset{\text{dual}}{\longleftrightarrow} \omega^i = \mathrm{d}x^i.$$

Let

$$\omega^i_j = \Gamma^i{}_{jk}\mathrm{d}x^k + \Pi^i_{jk}\mathrm{d}y^k, \tag{5.1}$$

is the 1-form on cotangent space of tangent bundle.

$$\begin{aligned}
0 = \mathrm{d}^2 x^i = \mathrm{d}\omega^i &\overset{(2.3)}{=} \omega^j \wedge \omega^i_j \\
&= \mathrm{d}x^j \wedge \left(\Gamma^i{}_{jk}\mathrm{d}x^k + \Pi^i_{jk}\mathrm{d}y^k\right) \\
&= \Gamma^i{}_{jk}\mathrm{d}x^j \wedge \mathrm{d}x^k + \Pi^i_{jk}\mathrm{d}x^j \wedge \mathrm{d}y^k \\
&= \sum_{j<k}\left(\Gamma^i{}_{jk} - \Gamma^i_{kj}\right)\mathrm{d}x^j \wedge \mathrm{d}x^k + \Pi^i_{jk}\mathrm{d}x^j \wedge \mathrm{d}y^k
\end{aligned}$$

thus, $\Pi^i_{jk} = 0$ and $\Gamma^i{}_{jk} = \Gamma^i_{kj} \Leftrightarrow$ torsion free. In order to determine $\omega^i_j$, now we only need to know the coefficients $\Gamma^i{}_{jk}$. Let $N^i_j = y^m\Gamma^i_{mj}$, plugging (5.1) into (2.4), one can get

$$\mathrm{d}g_{ij} = g_{im}\Gamma^m_{jl}\mathrm{d}x^l + g_{mj}\Gamma^m_{il}\mathrm{d}x^l + 2C_{ijm}\left(\mathrm{d}y^m + N^m_l\mathrm{d}x^l\right). \tag{2.7}$$

Note that

$$\begin{aligned}
\mathrm{d}g_{ij} &= \frac{\partial g_{ij}}{\partial x^l}\mathrm{d}x^l + \frac{\partial g_{ij}}{\partial y^l}\mathrm{d}y^l \\
&= \frac{\partial g_{ij}}{\partial x^l}\mathrm{d}x^l + 2C_{ijl}\mathrm{d}y^l,
\end{aligned}$$

then (2.7) implies [5](2.8)$\cdots \Rightarrow$[6](2.11).

Define

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^m_i\frac{\partial}{\partial y^m},$$

and note

$$C_{jlm} = \frac{\partial g_{jl}}{\partial y^m},$$

thus,

$$\frac{\delta g_{jl}}{\delta x^i} = \frac{\partial g_{jl}}{\partial x^i} - N^m_i\frac{\partial g_{jl}}{\partial y^m} = \frac{\partial g_{jl}}{\partial x^i} - 2N^m_i C_{jlm},$$

[5]the equation (2.8) is

$$\frac{\partial g_{ij}}{\partial x^l} = g_{im}\Gamma^m_{jl} + g_{mj}\Gamma^m_{il} + 2C_{ijm}N^m_l.$$

[6] the equation (2.11) is

$$\begin{aligned}
\Gamma^k_{ij} =& \frac{1}{2}g^{kl}\left\{\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right\} \\
&- g^{kl}\left\{C_{jml}N^m_i + C_{iml}N^m_j - C_{ijm}N^m_l\right\}.
\end{aligned}$$

then one can get

$$(2.11) \Leftrightarrow \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\delta g_{il}}{\delta x^j} + \frac{\delta g_{jl}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^l} \right\}$$

the above equation is quite important, there is a similar equation of Riemannian metric, in fact, in Rieman case, $\delta g_{ij}/\delta x^l = \partial g_{ij}/\partial x^l$, as $C_{ijl} = 0$.

Use $y^i$ to contract equation (2.11) one can get

$$N^k_j = \cdots \Rightarrow (2.12), \quad G^i := \frac{1}{2} N^i_j y^j = \frac{1}{2} \Gamma^i{}_{jk} y^j y^k,$$

use $\frac{1}{2} y^j$ to contract equation [7](2.12), one got [8](2.13), from which, we conclude that $G^i, N^i_j$ can be expressed by $g_{ij}$, the metric matrix of $F$. □

5.3. [9]**Chern Connection Continue.** In fact, suppose $\{e_i\}$ is the basis, and the dual basis of $\{e_i\}$ is $\left\{\omega^i\right\}$. Then there exist $\left\{\omega^i_j\right\}$ such that

$$\mathrm{d}\omega^i = \omega^j \wedge \omega^i_j,$$
$$\mathrm{d}g_{ij} = g_{kj}\omega^k_i + g_{ik}\omega^k_j + 2C_{ijk}\omega^{n+k},$$

where

$$\omega^{n+k} := \mathrm{d}y^k + y^i \omega^k_j.$$

Taking $\omega^i = \mathrm{d}x^i$, $\omega^i_j = \Gamma^i{}_{jk}\mathrm{d}x^k$. Then

$$N^i_j = \Gamma^i{}_{jk} y^k,$$
$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left\{ \frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{kl}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l} \right\}, \quad \Gamma^i_{jk} = \Gamma^i_{kj}$$
$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^r_k \frac{\partial}{\partial y^r}$$

particularly, if $(g_{ij})$ is a Riemannian metric, then $g_{ij}$ is independent of $y$, thus $\partial/\partial y^r = 0$.

$$G^i = \frac{1}{2} N^i_j y^j = \frac{1}{4} g^{il} \left\{ \frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{kl}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l} \right\} y^j y^k$$
$$= \frac{1}{4} \left\{ \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k.$$

[7]the equation (2.12) is

$$N^k_j = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\} y^i - 2g^{kl} C_{jml} G^m,$$

[8]the equation (2.13) is

$$G^i = \frac{1}{4} \left\{ \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k.$$

[9]Fourth Time of Lecture, Thursday, April 1, 2010

Further, use the homomorphism of $F$ with respect to $y$, we have

$$\left[F^2\right]_{x^l} = \frac{1}{2}\left[F^2\right]_{x^l y^i} y^i = \frac{1}{2}\left[F^2\right]_{x^l y^i y^j} y^i y^j = \frac{\partial g_{ij}}{\partial x^l} y^i y^j,$$
$$\left[F^2\right]_{x^k y^l} = \left[F^2\right]_{x^k y^l y^i} y^i = 2\frac{\partial g_{il}}{\partial x^k} y^i, \quad \left[F^2\right]_{x^k y^l} y^k = 2\frac{\partial g_{il}}{\partial x^k} y^i y^k,$$

and

$$G^i = \frac{1}{4} g^{il}\left\{\left[F^2\right]_{x^k x^l} y^k - \left[F^2\right]_{x^l}\right\} = {}^1 \cdots \Rightarrow (2.15)$$

which we called the decomposition of $G^i$.

Now let us set

$$G^i = P y^i + Q^i \tag{2.15}$$

where

$$P := \frac{F_{x^k} y^k}{2F},$$
$$Q^i := \frac{F}{2} g^{il}\left\{F_{x^k y^l} y^k - F_{x^l}\right\},$$

If $F$ is projectively flat, then $Q^i = 0$. $P$ is called the projective factor of metric $F$.

Set

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \in C^\infty(T(TM_0))$$

is a vector field on $TM_0$. Which satisfy

$$G^i(\lambda y) = \lambda^2 G^i(y)$$

called a *spray* on $(M, F)$. It has a integral curves, and whose projection to manifold $M$ is a geodesic.

Note that here we induce the spray by a metric, but in general, it can be defined without metric.

Next, we'll determine the coefficient of connection $N^i_j$. Differential (2.13) with respect to $y^j$ and use (2.12) (Note that Cartan tensor will appear in the differentiation of

[1]This can be get as follow:
note

$$\left[F^2\right]_{x^l} = 2FF_{x^l},$$
$$\left[F^2\right]_{x^k y^l} y^k = [2FF_{x^k}]_{y^l} y^k = 2F_{y^l} F_{x^k} y^k + 2FF_{x^k y^l} y^k$$
$$= 2\frac{1}{F} \cdot \frac{1}{2}\left[F^2\right]_{y^l y^m} y^m \cdot F_{x^k} y^k + 2FF_{x^k y^l} y^k$$
$$= \frac{2F_{x^k} y^k}{F} g_{ml} y^m + 2FF_{x^k y^l} y^k.$$

$g_{jl}$ with respect to $y^j$) we have

$$N^i_j = \frac{\partial G^i}{\partial y^j}.$$

In fact, we can get this by a directly calculate. Set $\partial g_{ij}/\partial x^k \overset{\text{def}}{=} \partial_{ijk}$.

As

$$G^i = \frac{1}{4} g^{il} \{\partial_{mlk} + \partial_{lkm} - \partial_{mkl}\} y^m y^k,$$

then

$$\begin{aligned}\frac{\partial G^i}{\partial y^j} = & \frac{\partial g^{il}}{\partial y^j}\frac{1}{4} \{\partial_{mlk} + \partial_{lkm} - \partial_{mkl}\} y^m y^k + \\ & \frac{1}{4} g^{il} \left\{\frac{\partial_{mlk}}{\partial y^j} + \frac{\partial_{lkm}}{\partial y^j} - \frac{\partial_{mkl}}{\partial y^j}\right\} y^m y^k + \\ & \frac{1}{4} g^{il} \{\partial_{mlk} + \partial_{lkm} - \partial_{mkl}\} \delta^m_j y^k + \\ & \frac{1}{4} g^{il} \{\partial_{mlk} + \partial_{lkm} - \partial_{mkl}\} y^m \delta^k_j\end{aligned}$$

note the following equations

- $g_{il} G^i = \frac{1}{4} \{\partial_{mlk} + \partial_{lkm} - \partial_{mkl}\} y^m y^k$;
- $\frac{\partial mlk}{\partial y^j} = \frac{\partial g_{ml}}{\partial x^k \partial y^j} = \frac{\partial C_{mlj}}{\partial x^k}$, $C_{mlj} y^m = 0 \Rightarrow \frac{\partial C_{mlj}}{\partial x^k} y^m = 0 \Rightarrow \frac{\partial_{mlk}}{\partial y^j} y^m = 0$, similarly, $\frac{\partial_{lkm}}{\partial y^j} y^k = 0$, $\frac{\partial_{mkl}}{\partial y^j} y^m = 0$;
- $g^{il} \left\{\partial_{jlk} + \partial_{lkj} - \partial_{jkl}\right\} y^k = g^{il} \left\{\partial_{mlj} + \partial_{ljm} - \partial_{mjl}\right\} y^m$;
- (2.12) $\Rightarrow$ $N^k_j = \frac{1}{2} g^{kl} \left\{\partial_{jli} + \partial_{lij} - \partial_{ijl}\right\} y^i - 2 g^{kl} C_{jml} G^m$, use a transformation ($i \to k, k \to i, l \to l, j \to j$), to get
$$\frac{1}{2} g^{il} \left\{\partial_{jlk} + \partial_{lkj} - \partial_{jkl}\right\} y^k = N^k_j + 2 g^{il} C_{jml} G^m.$$

we have

$$\begin{aligned}\frac{\partial G^i}{\partial y^j} &= C_{ilj} g_{ml} G^m + \frac{1}{2} g^{il} \left\{\partial_{jlk} + \partial_{lkj} - \partial_{jkl}\right\} y^k \\ &= C_{ilj} g_{ml} G^m + N^k_j + 2 g^{il} C_{jml} G^m.\end{aligned}$$

since

$$g^{il} g_{lm} = \delta^i_m,$$

we have

$$\frac{1}{2}\frac{\partial g^{il}}{\partial y^j} g_{lm} + \frac{1}{2} g^{il} \frac{\partial g_{lm}}{\partial y^j} = 0,$$

ie.,

$$2 g^{il} C_{jml} = 2 g^{il} C_{lmj} = -C_{ilj} g_{lm},$$

thus,

$$N^k_j = \frac{\partial G^i}{\partial y^j}.$$

5.4. **Landsberg Curvature.** Up to now, we have the following

$$G^i = \frac{1}{2}\Gamma^i{}_{jk}(x,y)y^jy^k,$$

see the following equation of (2.12), and

$$\frac{\partial G^i}{\partial y^j} = N^i_j = \Gamma^i{}_{jk}y^k,$$

$$\frac{\partial^2 G^i}{\partial y^j \partial y^k} = y^m \frac{\partial \Gamma^i_{jm}}{\partial y^k} + \Gamma^i{}_{jk}.$$

Let

$$L^i{}_{jk}(x,y) := \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \Gamma^i{}_{jk}(x,y) = y^m \frac{\partial \Gamma^i{}_{jm}}{\partial y^k}.$$

*Remark* 5.4. Note

$$\frac{\partial^2 G^i}{\partial y^j \partial y^k} \text{ and } \Gamma^i{}_{jk}(x,y)$$

is the coefficient of *Berwald connection* and *Chern connection* respectively.

Set

$$L_y := L^i{}_{jk}\omega^j \otimes \omega^k \otimes e_i,$$

called the *Landsberg tensor*, and

$$\mathcal{L} := \left\{L_y | y \in TM \setminus \{0\}\right\}$$

is called the *Landsberg curvature*, which is the first non-Riemannian geometric quantity. i.e., it equal to zero when the metric is a Riemannian metric, as the coefficient of Chern and Berwald connection both are zero.

*Remark* 5.5. For any Riemannian space, $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x)$ which is independent of $y$. Thus, $G^i = G^i(x,y) = \Gamma^i{}_{jk}y^iy^j/2$ is a quadrics function of $y$.

**Definition 5.2.**

- A Finsler metric $F$ on a manifold $M$ is called a *Berwald metric* if in any standard local coordinate system $(x^i, y^i)$ in $TM_0$, the Christoffel symbols $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x)$ are functions of $x \in M$ only, in which case, $G^i = \frac{1}{2}\Gamma^i{}_{jk}(x)y^jy^k$ are quadratic in $y = y^i \left.\frac{\partial}{\partial x^i}\right|_x$.
- $F$ is called a *Landsberg metric* if $L^i{}_{jk} = 0$.

Just by the definition, every Riemannian metric($g_{ij}(x,y) = g_{ij}(x) \Rightarrow C_{ijk} = 0 \Rightarrow \Gamma^i{}_{jk}(x,y) = \Gamma^i{}_{jk}(x)$) is a Berwald metric, and there exist non-Riemannian Berwald metrics, we will see this later. Further, note $L^i{}_{jk} = y^m \frac{\partial \Gamma^i{}_{mk}}{\partial y^j}$, it's easy to get $L^i{}_{jk} = 0$ for every Berwald metric, thus we have the following

**Proposition 5.1.** *Every Berwald metric is a Landsberg metric.*

*Remark* 5.6. The above proposition assert that every Berwald metric will be a Landsberg metric. The converse is a public open problem, ie., is a Landsberg metric must be a Berwald metric? In 2008, a canonical (i.e., $(g_{ij})$ is positively definite) Landsberg metric must be a Berwald metric has been proved by a American mathematician, and based on this result, Z.Shen have proved that if a $(\alpha, \beta)$ metric is canonical (positively definite), it must be a Berwald metric, this is a wonderful result, but later someone point out the former result is not true.

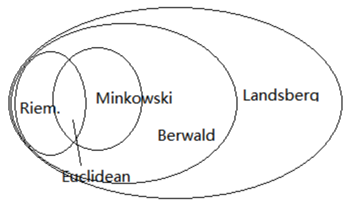


FIGURE 4. The Relation Between Metrics

5.4.1. *Calculate in Local Coordinate.* Suppose $(M, F)$ is a Finsler manifold, and $\{e_i(x)\}$ is a basis of $T_xM$, $\left\{\omega^i(x)\right\}$ is the dual basis to $\{e_i(x)\}$ in $T_x^*M$. Let $\left\{\omega_j^i\right\}$ denote the coefficients of Chern connection. Then

$$\nabla e_j = \omega_j^i \otimes e_i,$$

For any $\mathbf{X} \in C^\infty(\pi^* TM) \cong \Gamma(TM)$(sections of $TM$). Taking $\mathbf{X} = X^i e_i$,

$$\begin{aligned} \nabla \mathbf{X} &= \nabla(X^i e_i) = \mathrm{d}X^i \otimes e_i + X^j \nabla e_j \\ &= \mathrm{d}X^i \otimes e_i + X^j \omega_j^i \otimes e_i \\ &= \left(\mathrm{d}X^i + X^j \omega_j^i\right) \otimes e_i. \end{aligned}$$

Let $\mathbf{Y} \in C^\infty(TM \setminus \{0\})$, then

$$\nabla_{\mathbf{Y}} \mathbf{X} = \left\{\left(\mathrm{d}X^i\right)(\mathbf{Y}) + X^j \omega_j^i(\mathbf{Y})\right\} e_i,$$

where $\omega_j^i(\mathbf{Y}) \overset{\text{def}}{=} \langle \omega_j^i, \mathbf{Y} \rangle$.

Particularly, when $F$ is a Berwald metric, by definition, in any standard local coordinate system $(x^i, y^i)$ in $TM_0$, the Christoffel symbols $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x)$ are local function of $x \in M$ only. Let $\theta_j^i := \Gamma^i{}_{jk}(x)\mathrm{d}x^k$. Define

$$\mathrm{D}\mathbf{X} := \left\{\mathrm{d}\mathbf{X}^i + \mathbf{X}^j \theta_j^i\right\} \otimes \frac{\partial}{\partial x^i},$$

where $\mathbf{X} = X^j \frac{\partial}{\partial x^j} \in C^\infty(TM) := \Gamma(TM)$. It's easy the verify that D is a linear connection on $TM$, and $\mathrm{D}_{\mathbf{X}}\mathbf{Y} - \mathrm{D}_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$.

In fact, if $\mathbf{X} = X^j \frac{\partial}{\partial x^j}$, and $\mathbf{Y} = Y^j \frac{\partial}{\partial x^j}$, then

$$\begin{aligned} \mathrm{D}_{\mathbf{x}}\mathbf{Y} - \mathrm{D}_{\mathbf{Y}}\mathbf{X} &= \left\{\left(\mathrm{d}Y^i\right)(\mathbf{X}) + Y^j\theta^i_j(\mathbf{X})\right\}\frac{\partial}{\partial x^i} - \left\{\left(\mathrm{d}X^i\right)(\mathbf{Y}) + X^j\theta^i_j(\mathbf{Y})\right\}\frac{\partial}{\partial x^i} \\ &= \left\{\langle\frac{\partial Y^i}{\partial x^k}\mathrm{d}x^k, X^j\frac{\partial}{\partial x^j}\rangle + Y^j\langle\Gamma^i{}_{jk}(x)\mathrm{d}x^k, X^m\frac{\partial}{\partial x^m}\right\}\frac{\partial}{\partial x^i} \\ &\quad - \left\{\langle\frac{\partial X^i}{\partial x^k}\mathrm{d}x^k, Y^j\frac{\partial}{\partial x^j}\rangle + X^j\langle\Gamma^i{}_{jk}(x)\mathrm{d}x^k, Y^m\frac{\partial}{\partial x^m}\rangle\right\}\frac{\partial}{\partial x^i} \\ &= \left(X^j\frac{\partial Y^i}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial x^i} + \left(Y^j\Gamma^i{}_{jk}(x)X^k - X^j\Gamma^i{}_{jk}(x)Y^k\right)\frac{\partial}{\partial x^i} \\ &= \left(X^j\frac{\partial Y^i}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial x^i} \\ &= \mathbf{XY} - \mathbf{YX} = [\mathbf{X}, \mathbf{Y}]. \end{aligned}$$

It's quit the same as the case in Riemannian geometry, and the last condition is torsion-free. D is called the *Levi-Civita connection* of $F$.

Now, consider a Riemannian metric $g = g_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$ on $M$. Let $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x)$ denote the Christoffel symbols. Let D be the Levi-Civita connection defined as above with $\theta^i_j := \Gamma^i{}_{jk}(x)\mathrm{d}x^k$. One can easily verify that

$$\mathrm{d}\theta^i = \theta^j \wedge \theta^i_j, \quad \mathrm{d}g_{ij} = g_{ik}\theta^k_j + g_{jk}\theta^k_i. \tag{5.2}$$

are equivalent to

$$\mathrm{D}_{\mathbf{X}}\mathbf{Y} - \mathrm{D}_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}], \tag{5.3}$$

$$y[g(\mathbf{X}, \mathbf{Y})] = g(\mathrm{D}_y\mathbf{X}, \mathbf{Y}) + \mathrm{g}(\mathbf{X}, \mathrm{D}_y\mathbf{Y}), \tag{5.4}$$

where $y \in T_xM$ and $\mathbf{X}, \mathbf{Y} \in C^\infty(TM)$.

In fact, (5.2) are spacial cases of (2.3) and (2.4), since in Riemannian case, Cartan tensor will vanish. In order to verify the equivalence, first note that $\mathrm{D_y}\frac{\partial}{\partial \mathrm{x^j}} = \langle\theta^{\mathrm{i}}_{\mathrm{j}}, \mathrm{y}\rangle = \langle\Gamma^{\mathrm{i}}{}_{\mathrm{jk}}(\mathrm{x})\mathrm{dx^k}, \mathrm{y}\rangle$

## 6. Structure Equation

6.1. **Curvature.** The structure equation is the equations which depict the geometric quantities and their derivatives. Suppose $(M, F)$ is a Finsler manifold, and $\{e_i\}$ is the basis with the dual basis $\left\{\omega^i\right\}$. Then the Chern connection forms $\omega^i_j$ withe respect to $\{e_i\}$ are uniquely determined by

$$\begin{cases} \mathrm{d}\omega^i &= \omega^j \wedge \omega^i_j \qquad (2.29), \\ \mathrm{d}g_{ij} &= g_{kj}\omega^k_i + g_{ik}\omega^k_j + C_{ijk}\omega^{n+k} \qquad (2.30), \end{cases}$$

where

$$\omega^{n+k} = \mathrm{d}y^k + y^j\omega^k_j, \tag{2.31}$$

The curvature form,

$$\Omega^i_j = \mathrm{d}\omega^i_j - \omega^j \wedge \mathrm{d}\omega^i_j, \tag{2.32}$$

differentiate (2.29),

$$\begin{aligned} 0 = \mathrm{d}^2(\omega^i) &= \mathrm{d}\omega^j \wedge \omega^i_j - \omega^j \wedge \mathrm{d}\omega^i_j \\ &= (\omega^m \wedge \omega^j_m) \wedge \omega^i_j - \omega^j \wedge \left(\Omega^i_j + \omega^m_j \wedge \omega^i_m\right) \\ &= -\omega^j \wedge \Omega^i_j. \end{aligned} \tag{6.1}$$

Clearly, $\left\{\omega^i, \omega^{n+k}\right\}_{i=1}^n$ is a basis of $T^*_y(TM_0)$ (the cotangent bundle of tangent bundle). Note that $\Omega^i_j$ can be expressed in forms of $\omega^i \wedge \omega^j$, $\omega^i \wedge \omega^{n+j}$ and $\omega^{n+i} \wedge \omega^{n+j}$. By (6.1) the coefficient of $\omega^{n+i} \wedge \omega^{n+j}$ is 0. Write

$$\begin{aligned} \Omega^i_j &= \frac{1}{2} R_j{}^i{}_{kl}\omega^k \wedge \omega^l + P_j{}^i{}_{kl}\omega^k \wedge \omega^{n+l}, \\ \omega^j\Omega^i_j &= \frac{1}{2} R_j{}^i{}_{kl}\omega^j \wedge \omega^k \wedge \omega^l + P_j{}^i{}_{kl}\omega^j \wedge \omega^k \wedge \omega^{n+l} \\ &= \frac{1}{2}\sum_{j<k<l}\left(R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} - R_j{}^i{}_{lk} - R_l{}^i{}_{kj} - R_k{}^i{}_{jl}\right) \\ &= \sum_{j<k<l}\left(R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk}\right)\omega^j \wedge \omega^k \wedge \omega^l + \sum_{j<k}\left(P_j{}^i_{kl}\right). \end{aligned}$$

by the linear independent we can get (2.34), (2.35), (2.36). Let $\Omega^i = \mathrm{d}\omega^{n+i} - \omega^{n+j} \wedge \omega^i_j = y^j\Omega^i_j$, (see [1]). Write

$$\begin{aligned} \Omega^i &:= \frac{1}{2} R^{\,i}_{kl}\omega^k \wedge \omega^l - L^{\,i}_{kl}\omega^k \wedge \omega^{n+l} \\ R^{\,i}_{kl} &:= R_j{}^i{}_{kl}y^j \\ L^{\,i}_{kl} &:= -P_j{}^i{}_{kl}y^j \end{aligned}$$

then,

$$\begin{aligned} R^{\,i}_{kl} + R^{\,i}_{lk} &= 0, \\ R^{\,i}_{kl}y^k y^l &= 0, \end{aligned}$$

and by (2.36)

$$y^k L^{\,i}_{kl} = 0.$$

setting

$$R^i{}_k := R_j{}^i{}_{kl}y^j y^l = R^{\,i}_{kl}y^l$$

define

$$R_y := R^i{}_k \omega^k \otimes e^i,$$

is a liner map from $T_xM$ to $T_xM$, called Riemannian curvature. This definition is quite important, it can induce the definition of *flag curvature* and *Rich curvature*.

*Remark* 6.1. $R^i{}_k y^k = 0 \Leftrightarrow R_y(y) = 0.$

Write

$$R_{ij} = g_{ij} R^k{}_j \Rightarrow R_{ij} = Rji.$$

which will be proved later, but the case in Riemannian geometry is easy to prove.

6.1.1. *How to Determine $R_j{}^i{}_{kl}$, $P_j{}^i{}_{kl}$ ?* Let $\omega^i = \mathrm{d}x^i$, $\omega^{n+i} := \delta y^i = \mathrm{d}y^i + N^i_j \mathrm{d}x^j$.

$$\begin{aligned}
\omega^i_j &:= \Gamma^i jk \mathrm{d}x^k, \\
\Omega^i_j &:= \mathrm{d}\omega^i_j - \omega^k_j \wedge \omega^i_k \\
&= \frac{1}{2} R_j{}^i{}_{kl} \mathrm{d}x^k \wedge \mathrm{d}x^l + P_j{}^i{}_{kl} \mathrm{d}x^k \wedge \left(\mathrm{d}y^i + N^l_k \mathrm{d}x^k\right) \\
&= \left(\frac{\partial \Gamma^i{}_{jk}}{\partial x^l} \mathrm{d}x^l + \frac{\partial \Gamma^i{}_{jk}}{\partial y^l} \mathrm{d}y^l\right) \wedge \mathrm{d}x^k - \Gamma^m_{jk} \Gamma^i_{ml} \mathrm{d}x^k \wedge \mathrm{d}x^l.
\end{aligned}$$

$$\Rightarrow R_j{}^i{}_{kl} = -\frac{\partial \Gamma^i{}_{jk}}{\partial y^l} \tag{2.46}$$

Similarly,

$$\sum_{k<l} R_j{}^i{}_{kl} \mathrm{d}x^k \wedge \mathrm{d}x^l = \sum_{k<l} \left(\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i{}_{jk}}{\partial x^l} - \Gamma^s_{jk} \Gamma^i_{sl} + \Gamma^s_{jl} \Gamma^i_{sk}\right) - N^r_l P_j{}^i{}_{kr} \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

where,

$$\begin{aligned}
-N^r_i P_j{}^i{}_{kr} \mathrm{d}x^k \wedge \mathrm{d}x^l &= \sum_{k<l} \left(N^r_k P_j{}^i{}_{lr} - N^r_l P_j{}^i{}_{kr}\right) \mathrm{d}x^k \wedge \mathrm{d}x^l \\
&= \sum k < l \left(-N^r_k \frac{\partial \Gamma^i_{jl}}{\partial y^r} + N^r_l \frac{\partial \Gamma^i{}_{jk}}{\partial y^r}\right) \mathrm{d}x^k \wedge \mathrm{d}x^l.
\end{aligned}$$

use

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^r_k \frac{\partial}{\partial y^r}.$$

we can get (2.45). And contracting by $y^i$,

$$R^{\ i}_{\ kl} = R_j{}^i{}_{kl} y^j \cdots \Rightarrow (2.47) = \frac{\delta N^i_l}{\delta x^k} - \frac{\delta N^i_k}{\delta x^l}.$$

$$\Omega^i := \frac{1}{2} R^{\ i}_{\ kl} \omega^k \wedge \omega^l - L^{\ i}_{\ kl} \omega^k \wedge \omega^{n+l},$$

$$L^{\ i}_{\ kl} = -y^j P_j{}^i{}_{kl},$$

$$y^j \frac{\partial \Gamma^i{}_{jk}}{\partial y^l} = \frac{\partial \Gamma^i{}_{jk} y^j}{\partial y^l} - \Gamma^i_{kl}.$$

form (2.46) and $L^{\ i}_{\ kl} = y^j P_j{}^i{}_{kl}$, we have

$$L^{\ i}_{\ kl} = -y^j \frac{\partial \Gamma^i{}_{jk}}{\partial y^l} = \frac{\partial G^i}{\partial y^k \partial y^l} - \Gamma^i_{kl},$$

is the Landsberg curvature. Further, by $R^i_k = R_j{}^i{}_{jl} y^j y^l$, we get (2.49). Note $R^i_k$ (Riem. curvature) can be defined by spray.

## 7. Finsler Metric of Constant Flag Curvature

7.1. **Flag Curvature.** Suppose $(M, F)$ is a Finsler manifold, for any $y \in T_x M$, Let $P = \operatorname{span}\{y, u\}$, $u \in T_x M$. Define *flag curvature*

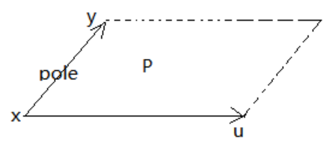


Figure 5. Flag

$$\mathcal{K}(p, y) := \frac{g_y\left(\mathbf{R}_y(u), u\right)}{g_y(y, y) g_y(u, u) - g_y(y, u)^2},$$

where

$$g_y\left(R^i_k u^i \frac{\partial}{\partial x^i}, u^j \frac{\partial}{\partial x^j}\right),$$

and $g_y$ is the inner product, $\mathbf{R}_y$ is the Riemannian curvature.

*Remark* 7.1. For Riemannian manifold flag curvature is the sectional curvature.

**Definition 7.1.**

$$h_{ij}(x, y) := g_{ij} - l_i l_j, \quad l_i = F_{y^i} = \frac{g_{ij} y^j}{F} \quad (l = \frac{y}{F(x, y)}),$$

$l_i$ is the i-th component of $l$, and $l_i = g_{ij} l^i$. $h_{ij}$ is called *angular metric tensor*.

Let $h^i_k = g^{ij}h_{jk}$, it is a (1, 1)-style tensor. Write $y = y^i \frac{\partial}{\partial x^i}$, $u = u^j \frac{\partial}{\partial x^j}$. Then,

$$\mathcal{K}(p, y) = \frac{R_{ij}(x, y)u^i u^j}{\left(F^2(x, y)g_{ij}(x, y) - g_{ik}g_{jl}y^k y^l\right) u^i u^j},$$

and $F$ is of constant flag curvature $\mu$ if and only if .

$$R_{ij} = \left(F^2 g_{ij} - y_i y_j\right)\mu, \quad y^j = g_{jl}y^l$$
$$\Leftrightarrow R_{ik} = \mu F^2 \left(\delta^i_k - l^i l^k\right) = \mu F^2 h_{ik} \Rightarrow (2.51).$$

$$R_j{}^i{}_{kl} = \mu \left(g_{jl}\delta^i_k - g_{ik}\delta^i_l\right),$$

so,

$$R_{jikl} = \mu \left(g_{jl}g_{ik} - g_{ik}g_{il}\right)$$
$$= -\mu \left(g_{jk}g_{il} - g_{jl}g_{ik}\right).$$

*Remark* 7.2 (Notes of theorem 2.3.2, see [1]). Suppose $(M, F)$ is a Finsler manifold, and is flat, i.e., $G^i = 0$. This is also equivalent to that it is locally Minkowski metric, or independent of $x$ on $U$, and by theorem 2.32, it is a Berwald metric. i.e., $\mathcal{B} = \mathcal{R} = 0$, where, $\mathcal{B}$ is the Berwald curvature, $B_j{}^i{}_{kl} := \partial^3 G^i / \left(\partial y^i \partial y^k \partial y^l\right)$. ($\mathcal{B} = 0$ means it is a Berwald metric)

*Remark* 7.3 (E.x.2.3.3, see [1]). $G^i = \Theta y^i = P y^i + Q^i$, $P = F_{x^k y^k} / (2F)$, $Q^i = F g^{il} \left\{F_{x^k x^l} y^k - F_{x^l}\right\} / 2 \Rightarrow G^i \neq 0, B_j{}^i{}_{kl} = \partial^3 G^i / \left(\partial y^j \partial y^k \partial y^l\right) \neq 0$. So, the metric in E.x.2.3.3 has flag curvature 0, but is not a Berwald metric.

## 8. [10]Bianchi Identities

In this lessen you should

- know the definition of covariant derivatives and how to calculate it,
- aware of some basic Bianchi Identities.

8.1. **Definitions of Covariant Derivatives.** Suppose $(M, F)$ is a Finsler manifold, let $\{e_i\} \subset \pi^* TM$ be basis and its dual basis is $\left\{\omega^i\right\}$. So, $\left\{\omega^i, \omega^n + i\right\}$ is a basis of $T^*(TM)$.

- *scaler function*. For any $f : TM \to \mathbf{R}$, define $\mathrm{d}f = f_{|i}\omega^i + f_{\cdot i}\omega^{n+i}$. In fact, we can get

$$f_{|i} = \frac{\partial}{\partial x^i} - N^r_i \frac{\partial f}{\partial y^r}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^r_i \frac{\partial}{\partial y^r}, \quad f_{\cdot i} = \frac{\partial f}{\partial y^i}.$$

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- *temsor.* Suppose $T = T_{ij}(x,y)\omega^i \otimes \omega^j$. Then,

$$\begin{aligned}(\nabla T)(e_i, e_j) &= \nabla T_{ij} - T\left(\nabla e_i, e_j\right) - T\left(e_i, \nabla e_j\right)\\ &= \mathrm{d}T_{ij} - T_{ik}\omega^k_j - T_{kj}\omega^k_i\\ &= T_{ij|k}\omega^k + T_{ij\cdot k}\omega^{n+k}.\\ \mathrm{d}T_{ij} &= T_{ij|k}\omega^k + T_{ij\cdot k}\omega^{n+k} + T_{ik}\omega^k_j + T_{kj}\omega^k_i.\end{aligned}$$

expand $\mathrm{d}T_{ij}$ and compere the corresponding coefficients,

$$\begin{aligned}T_{ij|k} &= \frac{\delta T_{ij}}{\delta x^k} - T_{sj}\Gamma^s_{jk} - T_{is}\Gamma^s_{jk}\\ &= \frac{\partial T_{ij}}{\partial x^k} - N^r_k\frac{\partial T_{ij}}{\partial y^r} - T_{sj}\Gamma^s_{ik} - T_{is}\Gamma^s_{jk}.\\ T_{ij\cdot k} &= \frac{\partial T_{ij}}{\partial y^k}.\end{aligned}$$

*Remark* 8.1 (the covariant derivatives of (1, 2)-style tensor).

$$T_{ij}{}^{k}{}_{l} = \frac{\partial \Gamma^k_{ij}}{\partial x^l} - N^r_i\frac{\Gamma^k_{ij}}{\partial y^r} - T^k_{rj}\Gamma^r_{il} - T^k_{ir}\Gamma^r_{jl} + T^r_{ij}\Gamma^k_{rl}.$$

*Remark* 8.2.
- $\mathrm{d}g_{ij} = g_{kj}\omega^k_i + g_{ik}\omega^k_j + 2C_{ijk}\omega^{n+k}$. So, $g_{ij|k} = 0$, $g_{ij\cdot k} = 2C_{ijk}$, which means this metric is not completely compatible.
- $\mathbf{Y} = y^i\frac{\partial}{\partial x^i}$, $y^i_{|j} = \frac{\delta y_i}{\delta x^j} + y^r\Gamma^i_{rj} = \frac{\partial y_i}{\partial x^j} - N^r_j\frac{\partial y^i}{\partial y^r} + N^i_j = -N^i_j + N^i_j = 0$. (note $\frac{\partial y_i}{\partial x^j} = 0$). And $y^i_{\cdot j} = \delta^i_j = \frac{\partial y^i}{\partial y^j}$.
- $F^2(x,y) = g_{ij}(x,y)y^iy^j$, $F^2_{|k} = g_{ij|k}y^iy^j = 0$, as $g_{ij|k} = 0$. Form which we can get

$$F_{|k} = 0.$$

8.2. **The Second Bianchi Identities.** we have

$$\begin{aligned}\Omega^i &:= \mathrm{d}\omega^{n+i} - \omega^{n+i} \wedge \omega^i_j \Rightarrow\\ \mathrm{d}\Omega^i &= -\mathrm{d}\omega^{n+i} \wedge \omega^i_j - \omega^{n+j} \wedge \mathrm{d}\omega^i_j\\ &= -\Omega^j \wedge \omega^i_j + \omega^{n+j} \wedge \Omega^i_j \qquad (2.60)\end{aligned}$$

On the other hand, by

$$\begin{aligned}\Omega^i &:= \frac{1}{2}R^{\ i}_{kl}\omega^k \wedge \omega^l - L^{\ i}_{kl}\omega^k \wedge \omega^{n+l} \Rightarrow\\ \mathrm{d}\Omega^i &= \frac{1}{2}\mathrm{d}R^{\ i}_{kl} \wedge \omega^k \wedge \omega^l + \frac{1}{2}R^{\ i}_{kl}\mathrm{d}\omega^k \wedge \omega^l\\ &\quad - \frac{1}{2}R^{\ i}_{kl}\omega^k \wedge \omega^l - \mathrm{d}L^{\ i}_{kl} \wedge \omega^k\omega^{n+l}\\ &\quad - L^{\ i}_{kl}\mathrm{d}\omega^k \wedge \omega^{n+l} + L^{\ i}_{kl}\omega^k \wedge \mathrm{d}\omega^{n+l}.\end{aligned}$$

Set

$$\begin{aligned}
\mathrm{I} &\stackrel{\text{def}}{=} \frac{1}{2}\mathrm{d}R^{\ i}_{kl} \wedge \omega^k \wedge \omega^l + \frac{1}{2}R^{\ i}_{kl}\mathrm{d}\omega^k \wedge \omega^l - \frac{1}{2}R^{\ i}_{kl}\omega^k \wedge \omega^l \\
&= \frac{1}{2}\left(R^{\ i}_{rl}\omega^r_k + R^{\ i}_{kr}\omega^r_l - R^{\ r}_{kl}\omega^i_r + R^i_{kl|r}\omega^r + R^i_{kl\cdot r}\omega^{n+r}\right) \wedge \omega^k \wedge \omega^l \\
&\quad + \frac{1}{2}R^{\ i}_{kl}\omega^r \wedge \omega^k_r \wedge \omega^l - \frac{1}{2}R^{\ i}_{kl}\omega^k \wedge \omega^r \wedge \omega^l_r \\
&= -\frac{1}{2}R^{\ r}_{kl}\omega^i_r\omega^k \wedge \omega^l + \frac{1}{2}R^{\ i}_{kl|r}\omega^r + R^{\ i}_{kl\cdot r}\omega^{n+r}.
\end{aligned}$$

and

$$\mathrm{II} \stackrel{\text{def}}{=} \mathrm{d}L^{\ i}_{kl} \wedge \omega^k \wedge \omega^{n+l} - L^{\ i}_{kl}\mathrm{d}\omega^k \wedge \omega^{n+l} + L^{\ i}_{kl}\omega^k \wedge \mathrm{d}\omega^{n+l}$$

by a similar calculate, we can express II as a linear combination of terms $\omega^i \wedge \omega^j \wedge \omega^k$, $\omega^i \wedge \omega^{n+j} \wedge \omega^{n+k}$, $\omega^i \wedge \omega^j \wedge \omega^{n+k}$. Thus, form which we can get (2.61), (2.62) and contract with $y^l$ we have (2.63).

*Remark* 8.3. Note, we can say Riemannian space is a non-color space, and Berwalt space have the same color everywhere, and a Finsler space is a colorful space.

## 8.3. **Relation Between The Curvature Tensor And The Finsler Metric.**

$$\mathrm{d}g_{ij} = g_{ik}\omega^k_j + g_{kj}\omega^k_i + 2C_{ijk}\omega^{n+k} \tag{2.30}$$

Note that,

$$\mathrm{d}C_{ijk} = C_{rjk}\omega^r_i + C_{irk}\omega^r_j + C_{ijr}\omega^r_k + C_{ijk|r}\omega^r + C_{ijk\cdot r}\omega^{n+r},$$

Differentiating (2.30),

$$0 = \mathrm{d}g_{ik} \wedge \omega^k_j + g_{ik}\mathrm{d}\omega^k_j + \mathrm{d}g_{kj} \wedge \omega^k_j + g_{kj}\mathrm{d}\omega^k_i + 2\mathrm{d}C_{ijk} \wedge \omega^{n+k} + C_{ijk}\mathrm{d}\omega^{n+k},$$

use the definition of $\Omega^k_i$,

$$\Omega^k_i = \mathrm{d}\omega^k_i - \omega^l_i \wedge^k_l,$$

we have,

$$0 = g_{ik}\Omega^k_j + g_{kl}\Omega^k_i + 2\left(C_{ijk|l}\omega^l + C_{ijk\cdot l}\omega^{n+l}\right) \wedge \omega^{n+k} + 2C_{ijk}\Omega^k,$$

it must be a combination of terms $\omega^i \wedge \omega^j$, $\omega^i \wedge \omega^{n+j}$, $\omega^{n+i} \wedge \omega^{n+j}$. So we can get (2.64), (2.65). Form (2.34), (2.35) and

$$\begin{cases} R_{jikl} + R_{jilk} = 0, \\ R_{jikl} + R_{kilj} + R_{lijk} = 0. \end{cases}$$

we have

$$\begin{aligned}
2\left(R_{klji} - R_{jikl}\right) &= \left(R_{klji} + R_{lkji}\right) - \left(R_{jikl} + R_{ijkl}\right) + \left(R_{kilj} + R_{iklj}\right) \\
&\quad + \left(R_{ljki} + R_{jlki}\right) + \left(R_{iljk} + R_{lijk}\right) + \left(R_{jkil} + R_{kjil}\right).
\end{aligned}$$

8.4. **Landsberg Curvature.** Recall that,

$$\mathcal{L} \stackrel{\text{def}}{=} L^{\ i}_{jk} \omega^i \otimes \omega^k \otimes e_i,$$

where

$$L^{\ i}_{jk} = -y^m P_m{}^i{}_{jk}, \quad P_m{}^i{}_{jk} = P_j{}^i{}_{mk}.$$

Set

$$L_{ijk} := g_{ir} L^r_{jk} = -y^m P_{mijk}, \quad P_{mijk} = P_{jimk},$$

Then,

$$L_{ijk} = \frac{1}{2} y^m P_{mijk} - \frac{1}{2} y^m P_{jimk} \xlongequal{\text{(2.65), and } C_{ijk}y^k=0} \frac{1}{2} y^m P_{imjk} + \frac{1}{2} y^m P_{ijmk} + C_{ijk|m} y^m$$

note by (2.65),

$$y^m P_{imjk} = L_{ijk}, \quad y^m P_{ijmk} = y^m p_{mjik} = -L_{jik} = -L_{ijk},$$

thus, $L_{ijk} = C_{ijk|m} y^m$ is the variance ratio of tensor along the geodesic.

8.5. **The Mean Landsberg Curvature.**

$$J_i = g^{jk} L_{ijk},$$

$$J_i = g^{jk} C_{ijk|m} y^m = I_{i|m} y^m.$$

you should verify the lemma 2.41, see [1], by yourself.

## **Part** 3. **Chapter 3, Geodesic**

### 9. SPRAY

9.1. **spray.** Suppose $(M, F)$ is a finsler manifold, On $TM_0$, define

$$G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

and for any $\lambda > 0$, $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$.

Let $\gamma = \gamma(t)$ is the integral curve of $G$ on $TM_0$ i.e.,

$$\dot{\gamma}(t) = G_{\gamma(t)}(\gamma(t), \dot{\gamma}(t)) \tag{3.1}$$

Assume that,

$$\gamma(t) = \left(x^i(t), y^i(t)\right)$$

then,

$$\dot{x}^i \frac{\partial}{\partial x^i} + \dot{y}^i \frac{\partial}{\partial y^i} = y^i \frac{\partial}{\partial} - 2G^i\left(x^i, y^i\right) \frac{\partial}{\partial y^i}$$

thus, $\dot{x}^i = y^i$, $\dot{y}^i = -2G^i\left(x^i, y^i\right)$, $x^i, y^i$ is a function of $t$.

Let $\sigma = \sigma(t)$ is the projection of $\gamma(t)$ on $M$ be the projection map $\pi$. i.e.,

$$\sigma(t) = \left(x^i(t)\right),$$

then we have

$$\ddot{x}^i(t) + 2G^i(x(t), \dot{x}(t)) = 0. \tag{3.2}$$

Conversely, if $\sigma = \sigma(t)$ satisfies (3.2), lift (by $\pi^{-1}$)

$$\gamma(t) = (\sigma(t), \dot{\sigma}(t)) \overset{\text{def}}{=} \dot{\sigma}(t).$$

form which we can get $\gamma(t)$ satisfies (3.1). The solution of equation (3.2) is called the *geodesic* of this spray. Suppose $(M, F)$ is a Finsler manifold, the *spray coefficients* or *geodesic coefficients* is

$$G^i(x,y) := \frac{g_{il}}{4}\left\{\left[F^2\right]_{x^k y^l} y^k - \left[F^2\right]_{x^l}\right\}.$$

A geodesic of a spray is called the geodesic of the metric.

*Example* 9.1.
- *Funk metric*. Suppose $\Omega$ is a convex domain in Euclidean space (for example, the indictrix of a Minkowski metric), then the solution of navigation problem is a Frunk metric. we already know that $\Theta_{x^k} = \Theta\Theta_{y^k}$, equally, $\Theta_{x^k y^l} y^k =$

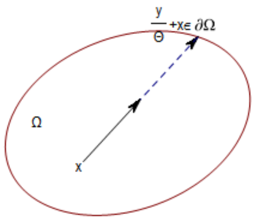



FIGURE 6. Frunk metric

  $\Theta_{x^l}$, we will know that this is a sufficient condition such that the metric is project flat in next lessen. Then,

$$G^i = \frac{1}{4} g^{il}\left\{\left[\Theta^2\right]_{x^k y^l} y^k - \left[\Theta^2\right]_{x^k}\right\} = \frac{1}{2}\Theta y^i.$$

- *Riemannian metric*.

$$F = \sqrt{a_{ij}(x) y^i y^j}, \quad G^i(x,y) = \frac{1}{2}\Gamma^i{}_{jk}(x) y j y^k,$$

  the geodesic (of this metric) is

$$\ddot{x}^i + \Gamma^i{}_{jk}(x(t))\dot{x}^j \dot{x}^k = 0$$

9.1.1. *$(\alpha, \beta)$-Metric.*

$$F = \alpha\phi(\frac{\beta}{\alpha}), \quad \phi = \phi(s) \in C^\infty,$$

if $\phi$ satisfy the following condition

$$\phi(s) > 0, \quad (\phi - s)\,\phi' + \left(b^2 - s^2\right)\phi'' > 0,$$

where $|s| \leq b < b_0$, $\|\beta\|_\alpha < b_0$, then we can prove that $F$ is a Finsler metric, called *$(\alpha,\beta)$-metric*. Let $\left\{e_i = \partial/\partial x^i\right\}$ is a basis, and the dual basis is $\theta^i = \mathrm{d}x^i$. Set

$$\theta^i_j = \bar{\Gamma}^j_{ik}(x)\mathrm{d}x^k,$$

is the Riemannian connection of $\alpha$. For $\beta = b_i(x)y^i$,

$$b_{i;j} = \frac{\delta b_i}{\delta x^j} - b_r\bar{\Gamma}^r_{ij} = \frac{\partial b_i}{\partial x^j} - b_r\bar{\Gamma}^r_{ij}.$$

Let

$$\gamma_{ij} = \frac{1}{2}\left(b_{i;j} + b_{j;i}\right), \quad s_{ij} = \frac{b_{i;j} - b_{j;i}}{},$$
$$\gamma^i_j = a^{ik}\gamma_{kj}, \quad s^i_j = a^{ik}s_{kj},$$
$$\gamma_j = b_i\gamma^i_j, \quad s_j = b_is^i_j = b^is_{ij}.$$

as $\beta = b_i\mathrm{d}x^i$,

$$\mathrm{d}\beta = \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right)\mathrm{d}x^i \wedge \mathrm{d}x^j,$$

thus, $\beta$ is *closed*, if and only if

$$\frac{\partial b_i}{\partial x^j} = \frac{\partial b_j}{\partial x^i} \Leftrightarrow s_{ij} = 0 \cdots \Rightarrow \text{Lemma 3.11, (see [1], p178).}$$

*Remark* 9.1. For any 1-form $\omega$, $\omega$ is closed if and only if it is *exact*, i.e., there exist an $\gamma \in C^\infty$, such that, $\mathrm{d}\gamma = \omega$.

9.1.2. *Randers metric.* Randers metric is a special metric, in fact, if we set $\phi = 1 + s$, then

$$F = \alpha + \beta = \alpha\phi(\alpha)\beta,$$
$$G^i = G^i_\alpha + Py^i + \alpha s^i{}_0.$$

where $P = \frac{e_{00}}{2F} - s_0$, $e_{ij} = \gamma_{ij} + b_is_j + b_js_i$, $e_{ij}y^iy^j = r_{00} + 2\beta s_0$. we know that *Randers metric* is a solution of the *navigation problem* on a Riemannian manifold. Here we'll give a classic method to induce the geodesic equation of Randers metric. First, we induce a $(\alpha,\beta)$-metric by the Riemannian metric, and the find the $\phi$, such that the Randers metric $F = \alpha\phi(\alpha/\beta)$.

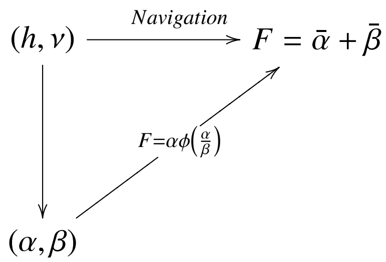
Suppose $h$ is the Riemannian metric, and $v$ is the direction, and the solution of the navigation problem is $F$, i.e., $h(x, y/F - v_x) = 1$. we know that is must be a Randers metric, so we can set it as $F = \bar{\alpha} + \bar{\beta}$, if we set $\lambda = 1 - \|v\|^2_h$, $\alpha = h/\sqrt{\lambda}$, $\beta = v_0/\lambda$, where $v_i = h_{ij}v^j$, and $v_0 = y^iv_i$. By a directly computation, we can set

$$\phi = \sqrt{1 + s^2} - s \Rightarrow F = \alpha\phi\left(\frac{\alpha}{\beta}\right) = \bar{\alpha} + \bar{\beta},$$

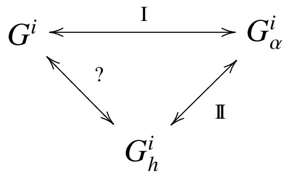
where,

$$\bar{\alpha} = \frac{\sqrt{\lambda h^2 + \nu_0}}{\lambda}, \quad \bar{\beta} = -\frac{\nu_0}{\lambda}.$$

the communicate diagram is




now, we'll use the diagram




where,

$$\mathrm{I} : G^i = G^i_\alpha - \alpha\phi s^i_0 + \frac{1}{2(1+b^2)\phi}\{2\alpha\phi s_0 + r_{00}\}\left\{\phi b^i - \frac{y^i}{\alpha}\right\},$$

and

$$\mathrm{II} : G^i_\alpha = \frac{1}{2}(\Gamma_\alpha)^i_{jk}\, y^j y^k,$$

where,

$$(\Gamma_\alpha)^i_{jk} = \frac{a^{il}}{2}\left\{\partial a_{jl}\partial x^k + \frac{\partial a_{kl}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^l}\right\}. \tag{1}$$

we know that

$$a_{ij} = \frac{h_{ij}}{\lambda}, \quad a^{ij} = \lambda h^{ij}, \quad b_i = -\frac{nu_i}{\lambda}, \quad b^i = \nu^i.$$

note

$$\frac{\partial a_{jl}}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\frac{h_{jl}}{\lambda}\right) = -\frac{\lambda_{|k} h_{jl}}{\lambda^2} + \frac{1}{\lambda}\frac{\partial h_{jl}}{\partial x^l}.$$

by the definition of $(h, \nu)$,

$$R_{ij} = \frac{1}{2}\left\{\nu_{i|j} + \nu_{j|i}\right\}, \quad s_{ij} := \frac{1}{2}\left\{\nu_{i|j} - \nu_{j|i}\right\},$$
$$R_j := \nu^i R_{ij}, \quad R := R_j \nu^j, \quad s_j := \nu^i s_{ij}.$$

the horizonal covariant derivative of $\lambda$ with respect to $h$ is

$$\lambda_{|k} = \left(1 - h_{ij}\nu^i\nu^j\right)_{|k} = -2h_{ij}\nu^i_{|k}\nu^j = -2\nu_{j|k}\nu^j = -2(R_{jk} + s_{jk})\nu^j = -2\left(R_k + s_k\right).$$

plugging it into (1), we can get

$$(\Gamma_\alpha)^i_{jk} = (\Gamma_h)^i_{|k} + \frac{1}{\lambda}\left\{(R_k + s_k)\,\delta^i_j + \left(R_j + s_j\right)\delta^i_k - \left(R^i + s^i\right)h_{jk}\right\},$$

so,

$$G^i_\alpha = \frac{1}{2}(\Gamma_\alpha)^i_{jk}\,y^j y^k = \cdots = G^i_h + \frac{1}{\lambda}(R_0 + s_0)\,y^i - \frac{1}{2\lambda}\left(R^i + s^i\right)h^2.$$

we should note that $h = h(y)$.

Furthermore, $s^i_0$, $s_0$, $r_{00}$ is determined by $b_{i|j}$, and finally is determined by the Christoffel symbols of $\alpha$. In fact,

$$\begin{aligned} b_{i|j} &= \frac{\partial b_i}{\partial x^j} - b_r\left(\bar{\Gamma}_\alpha\right)^r_{ij} \\ &= \frac{\partial}{\partial x^j}\left(\frac{v_i}{\lambda}\right) - b_r\left\{(R_h)^m_{ij} - \frac{1}{\lambda}(R_i + s_i)\,\delta^m_j + \left(R_j + s_j\right)\delta^m_i - (R^m + s^m)\,h_{ij}\right\}, \end{aligned}$$

at last,

$$\gamma_{ij} = \frac{1}{2}\left(b_{i;j} + b_{j;i}\right),$$

contract with $y^i, y^j$, we can get $r_{00}$. $s^i_j = a^{ih}s_k$, and $a^{ih}$ can be expressed by $h_{ij}$, so, we can get $s^i_j$ be expressed in terms of $h_{ij}$.

Note $b_i \cdot s^i_j = s_j$, form which we can induce lemma 3.13, see [1].

**Lemma 9.1** (lemma 3.1.3, see [1]). *For a Randers metric $F$ expressed in terms of a Riemannian metric $h$ and a vector field $V$ by (3.8[11]) , the spray coefficients $G^i$ of $F$ can be expressed in terms of the spray coefficients $G^i_h$ of $h$ and the covariant derivatives of $V$ with respect to $h$ as follows:*

$$G^i = G^i_h - FS^i_0 - \frac{1}{2}F^2(R^i + S^i) + \frac{1}{2}\left\{\frac{y^i}{F} - V^i\right\}\left\{2FR_0 - R_{00} - F^2R\right\}. \tag{9.1}$$

*Formula* (9.1) *is obtained by C.Robles in a different approach* [7].

## 10. [12]Shortest Path

### 10.1. **Geodesic.**

**Lemma 10.1** (3.2.1, see [1]). *Suppose $(M, F)$ is a Finsler manifold, $\sigma = \sigma(t)$ is a smooth geodesic, then $\|\dot\sigma(t)\|_F$ is a constant.*

*Proof.*

$$\begin{aligned} F^2(\sigma(t), \dot\sigma(t)) &= g_{ij}\left(\sigma(t), \dot\sigma(t)\right)\dot\sigma^i(t)\dot\sigma^j(t) \\ \frac{\mathrm{d}F^2\left(\sigma(t), \dot{\sigma(t)}\right)}{\mathrm{d}t} &= \frac{\partial g_{ij}}{\partial x^k}\dot\sigma^k\dot\sigma^i\dot\sigma^j + 0(\text{=contract of Cartan tensor}) + 2g_{ij}\ddot\sigma^i\dot\sigma^j, \end{aligned}$$

[11] $F = \frac{\sqrt{\lambda h^2 + V_0^2}}{\lambda} - \frac{V_0}{\lambda}$

[12] Sixth Time of Lecture, Thursday, April 15, 2010

and

$$
\begin{aligned}
0 = g_{ij|k} &= \frac{\partial g_{ij}}{\partial x^k} - N^r_k \frac{\partial g_{ij}}{\partial y^r} - g_{im}\Gamma^{\ m}_{jk} - g_{mj}\Gamma^{\ m}_{ik}, \\
\frac{g_{ij}}{\partial x^k} &= 2N^r_k C_{ijk} + g_{im}\Gamma^{\ m}_{jk} + g_{mj}\Gamma^{\ m}_{ik}, \\
\frac{\mathrm{d}\left[F^2(\sigma(t), \dot\sigma(t))\right]}{\mathrm{d}t} &= g_{im}N^m_k \dot\sigma^k \dot\sigma^i + g_{mj}N^m_k \dot\sigma^k \dot\sigma^j + 2g_{ij}\ddot\sigma^i \dot\sigma^j \\
&= 4g_{im}G^m \dot\sigma^i - 4g_{ij}G^i \dot\sigma^j \\
&= 0.
\end{aligned}
$$

so,

$$F(\sigma(t), \dot\sigma(t)) = \|\dot\sigma(t)\|^2_F = Const.$$

□

For any $p, q \in M$, $\mathrm{d}_F(p,q) = \inf_{c=\widetilde{pq}} \mathcal{L}_F(c)$. If $c\colon \sigma = \sigma(t)$ satisfies $\mathrm{d}_F(p,q) = \mathcal{L}_F(c)$, then $c$ is the *shortest path* through $p$ to $q$.

**Proposition 10.1** (proposition 3.2.2, see [1]). *For a shortest path in a Finsler manifold, any parametrization with constant speed is a smooth geodesic.*

*Proof.* **step 1.** Let $c = \widetilde{pq}, \sigma = \sigma(t), a \le t \le h$ is the shortest path. and $F(\sigma(t), \dot\sigma(t)) = \lambda = const.(> 0)$, variation of $c$:

$$H : [a,b] \times (-eps, eps) \to M$$

and satisfies $H(t,0) = \sigma(t)$, $H(a,s) = \sigma(a)$, $H(b,s) = \sigma(b)$, suppose $c_s$ is the variation curve, and expressed by $\sigma_s(t) = H(t,s)$, $a \le t \le b$. $V(t)$ is a vector field along $\sigma(t)$ satisfies $V(a) = V(b) = 0$, $V(t) = \frac{\partial H}{\partial s}(t,0)$ is the variation vector field. see figure 7. **step 2.**

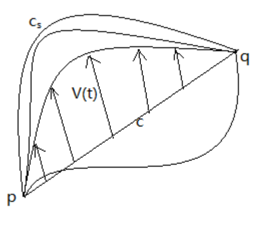



FIGURE 7. The Variation of Curve $c$ and The Variation Field

Let $\mathcal{L}(s) = \int_{c_s} F(\sigma_s, \dot\sigma_s)\,\mathrm{d}t$. So, $\mathcal{L}'(0) = 0$, we should show that

$$\ddot\sigma^i + 2G^i(\sigma, \dot\sigma) = 0.$$

As $\mathcal{L}(s) = \int_a^b F\left(H(t,s), \frac{\partial H}{\partial t}(t,s)\right)\mathrm{d}t$. If we set $x^k = H^k(t,s)$, $y^k = \frac{\partial H^k}{\partial x^k}(t,s)$. Note that

$$\frac{\partial y^k}{\partial s}|_{s=0} = \frac{\partial^2 H^k}{\partial t \partial s}(t,0) = \frac{\partial^2 H^k}{\partial s \partial t}(t,0) = \frac{\mathrm{d}V^k(t)}{\mathrm{d}t}.$$

and

$$\begin{aligned}\left[F^2\right]_{y^k} \frac{\mathrm{d}V^k(t)}{\mathrm{d}t} &= \left(\left[F^2\right]_{y^k} V^k\right)\Big|_t - \frac{\mathrm{d}\left[F^2\right]_{y^k}}{\mathrm{d}t} V^k \\ &= \left(\left[F^2\right]_{y^k} V^k\right)\Big|_t - \left[F^2\right]_{y^k x^l} \dot{\sigma}^l V^k - \left[F^2\right]_{y^k y^l} \ddot{\sigma}^l V^k,\end{aligned}$$

we have

$$\begin{aligned}\mathcal{L}'(0) &= \int_a^b \frac{1}{2F} \frac{\mathrm{d}\left[F^2\right]}{\mathrm{d}s}\Bigg|_{s=0} \mathrm{d}t \quad \text{F=}\lambda\text{, const.} \\ &= \int_a^b \frac{1}{2F} \left\{\left[F^2\right]_{x^k} V^k + \left[F^2\right]_{y^k} \frac{\mathrm{d}V^k(t)}{\mathrm{d}t}\right\}_{s=0} \mathrm{d}t \\ &= \int_a^b \frac{1}{2F} \left\{\left[F^2\right]_x^k - \left[F^2\right]_{y^k x^l} \dot{\sigma}^l - \left[F^2\right]_{y^k y^l} \ddot{\sigma}^l\right\} V^k \mathrm{d}t \\ &\quad + \int_a^b \frac{1}{2F}\Big|_{s=0} \frac{\mathrm{d}\left(\left[F^2\right]_{y^k} V^k\right)}{\mathrm{d}t}\mathrm{d}t.\end{aligned}$$

The last term of the above formula is equal to

$$\int_a^b \frac{1}{2F}\Big|_{s=0} \frac{\mathrm{d}\left(\left[F^2\right]_{y^k} V^k\right)}{\mathrm{d}t}\mathrm{d}t = \frac{1}{2F} \left[F^2\right]_{y^k} V^k\Big|_a^b.$$

At last, if we note that

$$g_{ij} = \frac{1}{2}\left[F^2\right]_{y^i y^j}, \quad G^j = \frac{1}{4} g^{jk} \left\{\left[F^2\right]_{x^l y^k} y^l - \left[F^2\right]_x^k\right\},$$
$$g_{jk} G^j = \frac{1}{4}\left\{\left[F^2\right]_{x^l y^k} y^l - \left[F^2\right]_{x^k}\right\}$$

we conclude

$$\mathcal{L}'(0) = \int_a^b \frac{1}{F} g_{jk}\left\{\ddot{\sigma}^l + 2G^l(\sigma, \dot{\sigma})\right\} V^k \mathrm{d}t + \frac{1}{F}\left(g_{jk}\sigma^j V^k\right)\Big|_a^b.$$

**step 3.** Take vector field $V(t)$ along $\sigma = \sigma(t)$,

$$V^i(t) = f(t)\left\{\ddot{\sigma}^i(t) + 2G^i(\sigma(t), \dot{\sigma}(t))\right\}$$

such that $f(a) = f(b) = 0$, $f(t) > 0$ whenever $a < t < b$. Finally, we get

$$\ddot{\sigma}^l + 2G^l(\sigma, \dot{\sigma}) = 0.$$

□

10.2. **Exponential Map.** An *exponential map* is a map form the tangent space to the underlying manifold. See figure 8 For any $y \in T_xM$, there exist geodesic $\sigma_y = \sigma(t)$, such

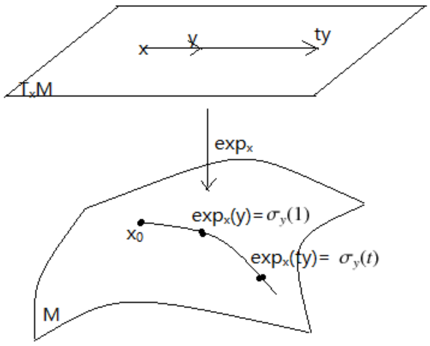



FIGURE 8. Exponential Map

that $\sigma_y(0) = x$, $\dot{\sigma}(0) = y$. Define $\exp_x(y) = \sigma_y(1)$, it can be proved that $||y||$ is equal to the length of geodesic form $x$ to $\sigma_y(1)$.

Let

$$\begin{array}{rccl} \gamma: & \left(-\frac{\varepsilon}{t}, \frac{\varepsilon}{t}\right) & \to & M \\ & s & \mapsto & \gamma(s) = \sigma_y(ts), \end{array}$$

$\gamma(s)$ is a another geodesic. $\gamma(0) = \sigma_y(0) = x$, $\gamma'(s) = t\dot{\sigma}_y(ts) \Rightarrow \gamma'(0) = ty$, this just means that $\gamma$ is a point on the geodesic which pass through $x$ and with the tangent vector $ty$. So $\gamma(s) = \sigma_y(ts) = \sigma_{ty}(s)$, then $\sigma_y(t) = \sigma_{ty}(1) = \exp_x(ty)$ .

## 11. PROJECTIVE EQUIVALENT FINSLER METRIC

**Definition 11.1.** Suppose $F, \bar{F}$ are metrics on $M$, we say $F$ and $\bar{F}$ are *projectively equivalent* if $F$ and $\bar{F}$ have the same geodesics as point set. Equivalently, for any geodesic $\sigma = \sigma(t)$ of $F$, there exist a transformation $t = t(\bar{t})$ such that $\bar{\sigma}(\bar{t}) = \sigma(t(\bar{t}))$ is a geodesic of $\bar{F}$ and vice versa.

**Definition 11.2.** Suppose $(M, F) \xleftrightarrow{\text{f is a diffeomorphism}} (\bar{M}, \bar{F})$, if $f^*\bar{F}$ and $F$ is projectively equivalent on $M$, then we call $F$ and $\bar{F}$ is projectively equivalent.

*Example* 11.1. Consider $S^2 \subset R^3$ as a submanifold, then on $S^2$ there is a metric induced by the inclusion map $\iota$, it can be proved they are projectively equivalent. see figure 9.

11.1. **Projectively equivalent Conditions.** Assume that $F, \bar{F}$ on $M$ are projectively equivalent. Now suppose $\sigma = \sigma(t)$ is a geodesic of $F$, satisfies $\sigma(0) = x$, $\dot{\sigma}(0) = y$, accordingly, $\bar{\sigma} = \bar{\sigma}(\bar{t})$ is a geodesic of $\bar{F}$, such that $\bar{t} = \bar{t}(t)$, $\sigma(t) = \bar{\sigma}(\bar{t}(t))$. Further more,

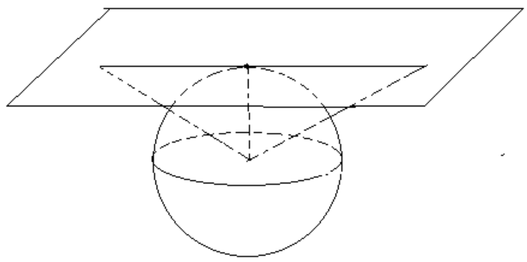


FIGURE 9. An Example of Projectively Equivalent

we can suppose $\bar{t}(0) = 0$, $\bar{t}'(0) = 1$. Then we have

$$\dot{\sigma}(0) = \dot{\bar{\sigma}}(0)\,(\bar{t}(0))\,\bar{t}'(0) = \dot{\bar{\sigma}}(0) = y,$$
$$\ddot{\sigma}(0) = (\dot{\bar{\sigma}}(\bar{t})\dot{\bar{t}}'(t)) = \ddot{\bar{\sigma}}(\bar{t})t'^2(0) + \dot{\bar{\sigma}}(\bar{t})t''(0) = \ddot{\bar{\sigma}}(0) + \dot{\bar{\sigma}}(0)\bar{t}''(0).$$

thus,

$$2G^i(x, y) = -\ddot{\sigma}^i(0) = -\ddot{\bar{\sigma}}^i(0) + \dot{\bar{\sigma}}^i(0)\bar{t}''(0)$$
$$= -2\bar{G}^i(x, y) - \bar{t}''(0)y^i$$

note $\bar{t}''(0)$ is $1(p)$-homogenous in y. So,

$$G^i(s, y) = \bar{G}^i(x, y) + P(x, y)y^i,$$

where,

$$P(x, y) = -\frac{1}{2}\bar{t}''(0),$$
$$P(x, \lambda y) = \lambda P(x, y), \quad \lambda > 0.$$

Conversely, if $G^i = \bar{G}^i + Py^i$. Consider geodesic of $\bar{F} : x = \bar{c}(\bar{t})$. Taking $\bar{t} = \bar{t}(t)$, such that,

$$\frac{dt}{d\bar{t}} > 0, \frac{d^2t}{d\bar{t}} = 2P\left(\bar{c}(\bar{t}), \dot{\bar{c}}(\bar{t})\frac{dt}{d\bar{t}}\right),$$

then, $c = \bar{c}\left(\bar{t}(t)\right) \hat{=} c(t)$ satisfies

$$\begin{aligned}
\ddot{c}^i(t) &= \frac{rd^2x^i}{dt^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{rdx^i}{\mathrm{d}\bar{t}}\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{rdx^i}{\mathrm{d}\bar{t}}\Big/\frac{\mathrm{d}t}{\mathrm{d}\bar{t}}\right) \\
&= \left(\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}\right)\frac{\mathrm{d}t}{\mathrm{d}\bar{t}} - \frac{\mathrm{d}x^i}{\mathrm{d}\bar{t}}\frac{\mathrm{d}^2t}{\mathrm{d}\bar{t}^2}\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}\right)\Big/\left(\frac{\mathrm{d}t}{\mathrm{d}\bar{t}}\right)^2 \\
&= \left(\ddot{\bar{c}}^i(\bar{t})\frac{\mathrm{d}t}{\mathrm{d}\bar{t}} - \dot{\bar{c}}^i(\bar{t})\frac{\mathrm{d}^2t}{\mathrm{d}t\bar{t}^2}\right)\Big/\left(\frac{\mathrm{d}t}{\mathrm{d}\bar{t}}\right)^3 \\
&= \left(-2\bar{G}^i\left(\bar{c}\left(\bar{t}\right),\dot{\bar{c}}\left(\bar{t}\right)\right)\frac{\mathrm{d}t}{\mathrm{d}\bar{t}} - \dot{\bar{c}}^i\left(\bar{t}\right)\cdot 2P\left(\bar{c}\left(\bar{t}\right),\dot{\bar{c}}\left(\bar{t}\right)\right)\frac{\mathrm{d}t}{\mathrm{d}\bar{t}}\right)\Big/\left(\frac{\mathrm{d}t}{\mathrm{d}\bar{t}}\right)^3 \\
&= \left(\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}\right)^2\left(-2\bar{G}^i\left(\bar{c},\dot{\bar{c}}\right) - 2P\left(\bar{c},\dot{\bar{c}}\right)\dot{\bar{c}}^i\right) \\
&= -2\bar{G}^i\left(c,\dot{c}\right) - P\left(c,\dot{c}\right)\dot{c}^i(t) \\
&= -2G^i(c,\dot{c}).
\end{aligned}$$

Finally, we have

$$\ddot{c}^i(t) + 2G^i\left(c(t),\dot{c}(t)\right) = 0.$$

The above proof can be found in [3]. Suppose $F, \bar{F}$ on $M$, and $G^i = \bar{G}^i + Py^i + Q^i$, $P = \frac{F_{;k}y^k}{2F}$, $Q^i = \frac{1}{2}Fy^{il}\left\{F_{;k\cdot l}y^k - F_{jl}\right\}$, where ";" is horizon variation derivative with respect to $\bar{F}$.

**Theorem 11.1** (theorem 3.3.1, see [1])**.** *Let $F$ and $\bar{F}$ be Finsler metrics on a manifold $M$. $F$ is projectively equivalent to $\bar{F}$ if and only if $F$ satisfies the following system,*

$$F_{;k\cdot l}y^k - F_{;l} = 0.$$

*Remark* 11.1. The following are some equivalent conditions of projectively equivalent metrics

- $F$ and $\bar{F}$ on $M$ are projectively equivalent;
- $$G^i = \bar{G}^i + Py^i, \quad P(x,\lambda y) = \lambda P(x,y), \forall\lambda > 0;$$
- $$F_{;k\cdot l}y^k - F_{;,l} = 0,$$

  in this case, $P = \frac{F_{;k}y^k}{2F}$, is called the *projective factor*. Note that the $F_{;k}$ is the horizon variation derivative with respect to $\bar{F}$, as the horizon variation derivative with respect to $F$ is zero.

*Remark* 11.2 (see [3]). Suppose $F$ and $\bar{F}$ on $M$. we call they are *trivial projectively equivalent* if one of the following condition satisfied

- $P = 0$;
- $G^i = \bar{G}^i$;

- $\bar{t} = a + tb$, i.e., it is a linear transformative. (use $P = -\frac{1}{2}\bar{t}''(0)$).

## 12. Projectively Flat Metric

Suppose $\bar{F}$ is a Euclidean or Minkowski metric, then they have the property: geodesic is a straight line. More precisely, any geodesic of $\bar{F}$ have the form $\bar{\sigma} = \bar{\sigma}(\bar{t}) = a + \bar{t}b$, where $a, b \in \mathbf{R^n}$ are constant vectors. then

**Definition 12.1.** we call a metric $F$ on $M$ is *projectively flat* if $F$ is projectively equivalent to $\bar{F}$.

**Lemma 12.1.** *the following condition is equivalent*

- *$F$ is a projectively flat metric;*
- *$\forall \bar{t} = f(t)$, $\sigma(t) = \bar{\sigma}(\bar{t}) = af(t) + b$ is a geodesic of $F$;*
- *$G^i = Py^i$, (because $G^i = 0 \Rightarrow \bar{N}^i_r = 0 \Rightarrow$ the vertical part of horizonal variation derivative$= 0$);*
- *$F_{x^k y^l} y^k - F_{x^l} = 0$, (G.Hamel, 1903). In this case, $P = \frac{F_{x^k} y^k}{2F}$.*

*Remark* 12.1 (Hilbert's Forth Problem: Distance (1900)). Suppose $U$ is an open set in $\mathbf{R}^n$, find the distance functions which satisfies that the shortest path is straight line between $p$ and $q$. So the projectively flat is just the smooth solution of *problem forth of Hilbert*.

*Remark* 12.2 (Bertrami theorem). Any Riemannian metric $H$ is projectively flat if and only if $H$ is of constant curvature.

*Example* 12.1. You should remember the result and compute or verify the example 3.4.2 – 3.4.7. see [1].

### 12.1. Projectively Flat Randers Metric.

**Proposition 12.1** (proposition 3.4.8, see [1])**.** *Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if the following condition satisfied*

- *$\alpha$ is locally projectively flat $\Leftrightarrow$ $\alpha$ is of constant curvature; and*
- *$\beta$ is closed, i.e., $\mathrm{d}\beta = 0 \Leftrightarrow s_{ij} = 0$. Note: $\beta$ is closed implies $F$ and $\bar{F}$ are projectively equivalent (Bacso-Matsumoto, 1997).*

*Example* 12.2. The example 3.4.9 of [1] is a classification of these metric have the property of projectively flat and constant curvature.

### 12.2. An Important Theorem on Projectively Flat Finsler Metric.

Recall the definition of *flag curvature*

$$\mathbb{K}(p, y) = \frac{g_y(\mathbf{R}_y(v), v)}{[F(x, y)]^2 g_y(v, v) - \left[g_y(y, v)\right]^2}.$$

If the metric $g$ is a Riemannian, then the $y$ in $g_y, \mathbf{R}_y$ is ineffective. If $\mathbb{K}(p, y) = \mathbb{K}(x, y)$, $y \in T_xM$, then we call $F$ if of *scalar flag curvature*.

**Theorem 12.1.** *If $F$ is projectively flat,*

$$G^i = Py^i, \quad P = \frac{F_{x^k} y^k}{2F},$$

*then, $F$ if of scalar flag curvature, and*

$$\mathbb{K} = \frac{P^2 - P_{x^k} y^k}{F^2}.$$

*which is* isotropic *on tangent bundle everywhere.*

## **Part 4.** [13]**Note of Chapter 4, Parallel Translations**

### 13. Parallel Vector Fields

Suppose $(M, F)$ is a Finsler manifold. $\nabla$ is the Chern connection. for any vector field $X \in X^i e_i$, especially we can set $e_i = \frac{\partial}{\partial x^i}$. we have

$$\begin{aligned}\nabla_v X &= \left\{ \mathrm{d}x^i(V) + X^j \omega_j^i(V) \right\} e_i \\ &= \left\{ \mathrm{d}X^i(V) +_X^j \Gamma_j{}^i{}_k \omega^k(V) \right\} e_i \\ &= \left\{ \mathrm{d}X^i(V) + X^j \Gamma^i{}_{jk} V^k \right\} e_i. \quad \text{Note } \Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x, y)\end{aligned}$$

### 14. Linearly Parallel

Let $c = c(t)$ is a smooth curve on $M$, $U = U^i(t) e_i$ is a vector field along $c$. Define (just substitute the $y$ in $\Gamma^i{}_{jk}$ by $\dot{c}$)

$$\mathrm{D}_{\dot{c}} := \nabla_{\dot{c}}^{\dot{c}}, \quad \dot{c} = \dot{c}^m(t) e_m.$$

by

$$\mathrm{D}_{\dot{c}} U(t) = \left\{ \mathrm{d}u^i(\dot{c}) + U^j \Gamma^i{}_{jk}(c, \dot{c}) \dot{c}^k \right\} \frac{\partial}{\partial x^i}.$$

*Remark* 14.1. As $\mathrm{d}U^i = \frac{\partial U^i}{\partial x^j} \mathrm{d}x^j$, $\mathrm{d}U^i(\dot{c}) = \frac{\partial U^i}{\partial x^j} \dot{c}^j = \frac{\mathrm{d}U^i}{\mathrm{d}t}$. we have

$$\begin{aligned}\mathrm{D}_{\dot{c}} U(t) &= \left\{ \dot{U}^i(t) + U^j(t) \Gamma^i{}_{jk}(c, \dot{c}) \dot{c}^k(t) \right\} \frac{\partial}{\partial x^i} \\ &= \left\{ \dot{U}^i(t) + U^j(t) N^i j(c, \dot{c}) \right\} \frac{\partial}{\partial x^i}\end{aligned}$$

from which we can get the (4.1), (4.2).

**Definition 14.1.** $U = U(t)$ is a vector field along curve $c$, we call it is a *linearly parallel vector field* if $\mathrm{D}_{\dot{c}} U(t) = 0$, or equally, $\dot{U}^i(t) + U^j(t) N^i_j(c, \dot{c}) = 0$.

[13]Seventh Time of Lecture, Thursday, April 22, 2010

*Remark* 14.2. In $\mathbf{R^n}$, $D_{\dot c}U(t) = \dot U(t) = 0 \Leftrightarrow U(t) = \text{const.}$ i.e., with the same size and direction, but different position.

Let $\mathcal{T} = T_{ij}(x,y)dx^i \otimes dx^j$ on $TM_0$. Recall that $(\nabla T)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \nabla T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) - T\left(\nabla\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) - T\left(\frac{\partial}{\partial x^i}, \nabla\frac{\partial}{\partial x^j}\right) \equiv dT_{ij} - T_{ik}\omega_j^k - T_{kj\omega_i^k}$, which is an 1-form, so is equal to $T_{ij|k}dx^k + T_{ij\cdot k}\delta y^k$ too. You should remember that $\delta y^k = \omega^{n+k} = dy^k + y^j\omega_j^k = dy^k + N_l^k(x,y)dx^l$, and $T_{ij|k} = \frac{\partial T_{ij}}{\partial x^k} - N_k^r\frac{\partial T_Pij}{\partial y^r} - T_{mj}\Gamma_{ik}^m - T_{im}\Gamma_{jk}^m$.

**Definition 14.2.**

$$\begin{aligned} T_y: \quad T_xM \times T_xM &\to \mathbf{R} \\ T_y(u,v) &= T_{ij}(x,y)u^iv^j \end{aligned}$$

where $y \in TM, y \neq 0$ and $u = u^i\frac{\partial}{\partial x^i}, v = v^j\frac{\partial}{\partial x^j}$.

Let $\sigma = \sigma(t)$ is a geodesic. $u = u(t), v = v(t)$ are linearly parallel vector field along $\sigma$. Put

$$T(t) = T_{\dot\sigma(t)}(u(t), v(t)) = T_{ij}(\sigma(t), \dot\sigma(t))u^i(t)v(^j(t).$$

Then, note $\sigma$ is a geodesic, and followed by use of the fact $u(t), v(t)$ both are linearly parallel vector field, we have

$$\begin{aligned} T'(t) &= \frac{\partial T_{ij}}{\partial x^k}\dot\sigma u^iv^j + \boxed{\frac{\partial T_{ij}}{\partial y^k}\ddot\sigma^k u^iv^j} + T_{ij}(\sigma,\dot\sigma)\dot u^iv^j + T_{ij}(\sigma,\dot\sigma)u^i\dot v^j \\ &= \frac{\partial T_{ij}}{\partial x^k}\dot\sigma u^iv^j - \boxed{2G^k(\sigma,\dot\sigma)\frac{\partial T_{ij}}{\partial y^k}u^iv^j} + T_{ij}(\sigma,\dot\sigma)\dot u^iv^j + T_{ij}(\sigma,\dot\sigma)u^i\dot v^j \\ &= \frac{\partial T_{ij}}{\partial x^k}\dot\sigma u^iv^j - 2G^k(\sigma,\dot\sigma)\frac{\partial T_{ij}}{\partial y^k}u^iv^j - T_{ij}v^mN_m^i(\sigma,\dot\sigma)v^j - T_{ij}u^iv^mN_m^j(\sigma,\dot\sigma) \\ &= \left\{\frac{\partial T_{ij}}{\partial x^k}\dot\sigma^k - 2G^k(\sigma,\dot\sigma)\frac{\partial T_{ij}}{\partial y^k} - T_{mj}N_i^m(\sigma,\dot\sigma) - T_{im}N_j^m\right\}u^iv^j \end{aligned}$$

Compare it with

$$T_{ij|k} = \frac{T_{ij}}{\partial x^k} - N_k^r\frac{\partial T_{ij}}{\partial y^r} - T_{mj}\Gamma_{ik}^m - T_{im}\Gamma_{jk}^m,$$

by set $x = \sigma(t), y = \dot\sigma(t)$ in $\Gamma$ of above formula. we can conclude that

$$T'(t) = T_{ij|k}(\sigma,\dot\sigma)\dot\sigma^k u^iv^j.$$

Just apply the above argument, we induce the following

**Lemma 14.1** (lemma 4.1.1, see [1])**.** *Let $\sigma = \sigma(t)$ be a geodesic in a Finsler manifold $(M,F)$ and let $u = u(t), v = v(t)$ be linearly parallel vector fields along $\sigma$. Then for the family of* induced inner product $g_{\dot\sigma}$ *along* $\sigma$,

$$g_{\dot\sigma(t)}(u(t), v(t)) = \text{const.}$$

14.1. **Parallel.** Let $c = c(t)$ is a $C^\infty$ curve on $M$. $u = u(t)$ is a vector field along $c$. Define (substitute $y$ in $\Gamma^k_{ij}(x,y)$ with $u(t)$)

$$\nabla_{\dot c} u(t) := \nabla^u_{\dot c} u(t),$$

recall that

$$\nabla_v u = \left\{ \left( \mathrm{d}u^i \right) v + u^j \Gamma^i{}_{jk}(x,y) v^k \right\} \frac{\partial}{\partial x^i},$$

so,

$$\nabla_{\dot c} u(t) = \left\{ \dot u^i(t) + N^i_j(c(t), u(t)) \dot c^j(t) \right\} \frac{\partial}{\partial x^i}.$$

**Definition 14.3.** we call $u = u(t)$ is *parallel along* $c$, $c = c(t)$ if $\nabla^u_{\dot c} u(t) = 0$, or equally, $\dot u(t) + N^i_j(c(t), u(t)) \dot c^j(t)$.

*Remark* 14.3. A compare of the two kinds of parallel: if $c = c(t)$ is a geodesic, then

$$\mathrm{D}_{\dot c} u = 0 \Leftrightarrow \nabla^c_{\dot c} u(t) = \dot u^i(t) + u^j N^i_j(c(t), \dot c(t)) = 0,$$
$$\nabla_{\dot c} u = 0 \Leftrightarrow \dot u^i(t) + N^i_j(c(t), u(t)) \dot c^j(t) = 0.$$

?? so, the vector field of a geodesic $c$ is the parallel vector field along itself.

**Lemma 14.2** (Lemma 4.1.2, see [1])**.** *Let $c = c(t)$ is a curve, $u = u(t)$ is the parallel vector field along $c$, then*

$$F\left(c(t), u(t)\right) = \mathrm{const.}$$

*Proof.*

$$0 = g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k} - N^r_k \frac{\partial g_{ij}}{\partial y^r} - g_{mj} \Gamma^m_{ik} - g_{im} \Gamma^m_{jk}$$
$$= \frac{\partial g_{ij}}{\partial x^k} - 2C_{ijm} N^m_k - g_{mj} \Gamma^m_{ik} - g_{im} \Gamma^m_{jk}.$$

□

*Remark* 14.4. This parallel has a poorer condition compare with the linearly parallel, but have a better property.

14.2. **Affinely equivalence.**

**Definition 14.4.** Two Finsler metrics $F$ and $\bar F$ on a manifold $M$ are side to be *affinely equivalent* if they have the same geodesics as parameterized curves. That is

$$G^i = \bar G^i \quad \text{(or equally, } P = 0\text{)}.$$

In general,

$$G^i = \bar G^i + P y^i + Q^i,$$

where $P = \frac{F_{;k}y^k}{2F}$, $Q^i = \frac{1}{2}Fg^{il}\left\{F_{;k\cdot l}y^k - F_{;l}\right\}$. So we have

$$y_iQ^i = 0, \quad y_i = g_{ij}y^j \Leftarrow \text{“}G^i = \bar{G}^i \Leftrightarrow Py^i + Q^i = 0\text{”},$$

$$Py_iy^i = 0 \Leftarrow PF^2 = 0 \Leftarrow P = 0, Q^i = 0 \Leftarrow F_{;k} = 0.$$

In fact we have the following

**Theorem 14.1** (Theorem 4.1.3, see [1])**.** *$F, \bar{F}$ on $M$ are affinely equivalent if and only if $F_{;k} = 0$.*

*Example* 14.1. Note that $F_{;k}$ can viewed as the horizonal covariant about Euclidean metric.

**Lemma 14.3** (Lemma 4.1.5, see [1])**.** *$F, \bar{F}$ on $M$ are affinely equivalent if and only if for any $\bar{F}$-parallel vector field $u = u(t)$ along any curve $c = c(t)$, we have*

$$F\left(c(t), u(t)\right) = \text{const.}$$

## 15. [14]Parallel Translations

We have defined two kind of parallel: $D_{\dot{c}} = \nabla_{\dot{c}}^{\dot{c}}$ and $\nabla_{\dot{c}}(U(t)) = \nabla_{\dot{c}}^{U}U(t)$, where $U(t)$ satisfy $\dot{U}^i(t) + \dot{c}^j(t)N_j^i(c(t), U(t)) = 0$.

15.1. **A translation from $T_pM$ to $T_qM$.** Suppose $(M, F)$ is a Finsler manifold, for any $c^\infty$ curve (there and there after $c^\infty$ may be piecewise smooth) $c(t)$, $a \le t \le b$, $p = c(a)$, $q = c(b)$. Define

$$\begin{array}{rccc} P_c\colon & T_pM & \to & T_qM \\ & U & \mapsto & U(b) \end{array}$$

where $U = U(t)$ is a parallel vector field along $c$.

*Remark* 15.1.
- $P_c(\lambda U) = \lambda U(b) = \lambda P_c(u)$,
- $P_c$ is nonlinear, i.e., if $U(t)$, $V(t)$ are parallel vector fields along $c$, is not implied that $U(t) + V(t)$ is parallel along $c$.

15.2. **Holonomy group.** Suppose $P \in M$, $C_p = \{c\colon [0,1] \to M | c(0) = c(1) = p, c \in c^\infty\}$. Next we'll define a "product" on $C_p$ such that $C_p$ is a group under this product.

- For any $c_1, c_2 \in C_p$, define

$$c_1 * c_2 = \begin{cases} c_2(2t), & \text{if } 0 \le t \le 1/2, \\ c_1(1-2t), & \text{if } 1/2 \le t \le 1. \end{cases}$$

  $p_{c_1} * p_{c_2} = p_{c_1} \circ p_{c_2} \colon T_pM \to T_pM.$

[14]Eighth Time of Lecture, Thursday, April 29, 2010

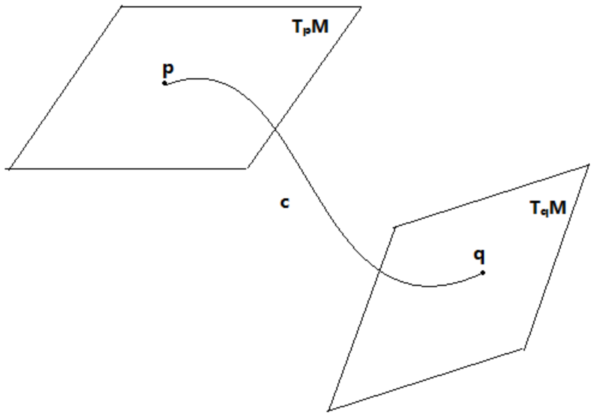



FIGURE 10. The Translation between Tangent Spaces

- For any $c \in C_p$, define $c_-(t)\colon c(1-t)$, $0 \le t \le 1$. then, note

$$c * c_-(t) = \begin{cases} c_-(2t), & \text{if } 0 \le t \le 1/2, \\ c(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

by the definition of $c_-$, we have

$$c * c_-(t) = \begin{cases} c(1-2t), & \text{if } 0 \le t \le 1/2, \\ c(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

which is the *identity of* $C_p$.

If $U = U(t)$ is a parallel vector field along $c$, define $U_-(t) = U(1-t)$, $0 \le t \le 1$. Then, note

- $\dot{U}^i_-(t) = -\dot{U}^i(t)$, $\dot{c}^i_-(t) = -\dot{c}^i(t)$.
- $\dot{U}^i(t) + \dot{c}^j(t)N^i_j(c(t), U(t)) = 0$, which means $U$ is the parallel vector field along $c$.

we have

$$-\dot{U}^i(1-t) + \left(-\dot{c}^j(1-t)N^i_j(c(1-t), U(1-t))\right) = 0$$
$$\dot{U}^i_-(t) + \dot{c}^j_-(t)N^i_j(c_-(t), U_-(t)) = 0$$

which just means $U_-(t)$ is a parallel vector field along $c$. what's more

$$U(0) = U_-(1) = P_{c_-}U_-(0) = P_{c_-}U(1) = P_{c_-} \circ P_c(U) = P_{c_- * c}(U) = \text{identity}.$$

Let $H_p = \left\{P_c\colon T_pM \to T_pM | c \in C_p\right\}$. Then $H_p$ is a group, called *holonomy group*

Use a holonomy group we can construct a Finsler metric which are affinely equivalent to a given Finsler metric.

**Proposition 15.1** (Proposition 4.2.2). *Let $(M, \bar{F})$ is a Finsler manifold,*

- *$H_p$ is the holonomy group of $\bar{F}$ at $p$,*
- *$F_p$ is the Minkowski norm on $T_pM$, and is $H_p$* invariant. *i.e., $F_p(U) = F_p(P_cU)$, for any $U \in T_pM$, $c \in C_p$.*

*then, $F_p$ can be extended to a Finsler metric $F$ on $M$, such that $F$ is* affinely equivalent *to $\bar{F}$.*

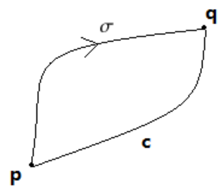



FIGURE 11. $\tau = c^{-1} \circ \sigma \in C_p$

*Proof.* **step 1.** For any $q \in M$, there exist a $C^\infty$ curve $c = c(t)$ such that $c(0) = p$, $c(1) = q$. Define $F(q)\colon T_qM \to [0, +\infty)$, $F_q(P_c(V)) \overset{\text{def}}{=} F_p(V)$, where $V \in T_pM$, and note $P_c \overset{\text{def}}{=} T_pM \to T_qM$, $V \mapsto P_c(V)$.
**step 2.** For any $\sigma = \sigma(t)$, $t \in [0, 1]$ and $\sigma(0) = p$, $\sigma(1) = q$. Let $\tau\colon c^{-1} * \sigma \in C_p$, $P_\sigma = P_c * c^{-1} \circ P_\sigma = P_{c*c^{-1}} * \sigma = P_c \circ P_\tau$. Then, for any $V \in T_pM$, $F_q(P_\sigma(V)) = F_q(P_c \circ P_\tau(V)) = F_q(P_c(P_\tau(V))) = F_p(P_\tau(V)) = F_p(V) = F_q(P_c(V))$. the last equation is obtained by $P_\tau \in H_p$.
**step 3.** At last we'll show that $F, \bar{F}$ is affinely equivalent.

For any $c = c(t)$, and $U = U(t)$ is a vector field which is parallel along $c$. There $a \le t \le b$, $c(a) = p$, $U(a) = u$. Put $q = c(t)$, then $F_q(U(t)) = F_{c(t)}(U(t)) = F_q(P_c(u)) = F_p(u)$ is constant. Use lemma 4.1.5 (see [1]) we finished the proof. □

## 16. BERWALD METRICS

16.1. **Recall.** a Berwald metric $F$ on $M$ is defined by $G^i$ (the *geodesic coefficient*) is quadratic polynomial, i.e.,

$$G^i = \frac{1}{2}\Gamma^i{}_{jk}(x)y^jy^k,$$

where

$$\Gamma^i{}_{jk} = \frac{1}{2}g^{il}\left\{\frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l}\right\}.$$

Let

$$L^i{}_{jk} = \Gamma^i{}_{jk} - G^i_{jk} = y^l\frac{\partial\Gamma^i_{lj}}{\partial y^k}$$

where

$$G^i_{jk} = G^i_{y^jy^k}, \quad \text{is the } \textit{coefficient of Berwald connection}.$$

Then $F$ is a Berwald metric $\Leftrightarrow \Gamma^i{}_{jk}(x,y) = \Gamma^i{}_{jk}(x) \Leftrightarrow L^i{}_{jk} = 0, \Gamma^i{}_{jk}(x) = G^i_{jk}(x) \Leftrightarrow B^i_{jkl} = 0$. where $B^i_{jkl} = \partial^3 G^i/(\partial y^j \partial y^k \partial y^l)$ is the *Berwald curvature*.

*Remark* 16.1 (curvature). chern: $\Omega^i_j = \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l + P^i_{jkl}\omega^k \wedge \omega^{n+l}$,
Berwald: $\Omega^i_j = \frac{1}{2}K^i_{jkl}\omega^k \wedge \omega^l - B^i_{jkl}\omega^k \wedge \omega^{n+l}$.

*Remark* 16.2. For Berwald metric, $\Gamma^i{}_{jk}(x,y) = \Gamma^i{}_{jk}(x) \Rightarrow N^i_j(x,y) = \Gamma^i{}_{jk}(x)y^k$.

For any $c = c(t) \in c^\infty$, $a \le t \le b$. The two parallel properties:

- linearly parallel: $U = U(t)$ is linearly parallel along $c \Leftrightarrow \dot{U}^i(t) + U^j(t)N^i_j(c(t), \dot{c}(t)) = 0$, i.e., $\dot{U}^i(t) + \dot{U}^j(t)\Gamma^i{}_{jk}(c(t))\,\dot{c}^k(t) = 0$.
- non-linearly parallel: $U = U(t)$ is non-linearly parallel along $c \Leftrightarrow \dot{U}^i(t) + \dot{c}^j(t)N^i_j(c(t), U(t)) = 0$, i.e., $\dot{U}^i(t) + \dot{c}^j(t)\Gamma^i{}_{jk}(c(t))\,U^k(t) = 0$.

So, for Berwald metric, there is only one parallel, or $U(t)$ is linearly parallel if and only if $U(t)$ is parallel along $c$.

**Proposition 16.1** (Proposition 4.3.2, see [1]). *Let $(M,F)$ be a Berwald metric, then for any $c = c(t) \in c^\infty$, $0 \le t \le 1$, $p = c(0)$, $q = c(1)$, we have*

$$P_c : \quad T_pM \quad \to \quad T_qM$$

*is a* linear isometric.

16.2. **The classification of Berwald metrics.**

**Proposition 16.2** (Proposition 4.3.3, see [1]). *A Finsler metric $F$ on $M$ is Berwald metric if and only if there is a Riemannian metric $\alpha$ such that $F$ is affinely equivalent to $\alpha$, i.e., $G^i_F = G^i_\alpha$.*

**Theorem 16.1** (theorem 4.3.4, local structure theorem, see [1], Z.I.Szobo, Tensor, 1981, 35). *Let $(M,F)$ be a Berwald manifold. Then $(M,F)$ can be locally decomposed to a product of locally Minkowski manifolds, Riemannian manifolds and locally irreducible locally symmetric Berwald manifolds of rank $r \ge 2$.*

**Corollary 16.1** (corollary 4.3.5, see [1]). *Any two-dimensional Berwald manifold is either locally Minkowskian or Riemannian.*

## 17. Landsberg metrics

Recall that $L^i{}_{jk} = \Gamma^i{}_{jk}(x,y) - G^i_{jk}(x,y)$ $\boxed{\text{note, } G^i_{jk} = G^i_{y^jy^k}}$ $= y^l \frac{\partial \Gamma^i_{lk}}{\partial y^j}$. $F$ is *Landsberg metric* if $L^i{}_{jk}(x,y) = 0$, or equally, $\Gamma^i{}_{jk}(x,y) = G^i_{jk}(x,y)$. Recall that $B^i_{jkl} = 0$ if and only if $G^i_{jk} = G^i_{jk}(x)$. i.e.,

$$\begin{array}{ccc} \Gamma^i{}_{jk}(x,y) & \overset{\text{Landsbergmetric}}{=\!=\!=\!=} & G^i_{jk}(x,y) \\ \uparrow & & \uparrow \\ B^i_{jkl} = 0 & \overset{\text{BerwaldMetric}}{\Longleftrightarrow} & G^i_{jk} = G^i_{jk}(x) \end{array}$$

we know that

$$B^i_{jkl} = 0 \Leftarrow \Gamma^i{}_{jk}(x,y) = \Gamma^i{}_{jk}(x,y) \Leftarrow L^i{}_{jk}(x,y) = 0,$$

so every *Berwald metric* is a *Landsberg metric*, but the converse problem is an open problem, i.e., asked that wether there exist a regularly none-Landsberg metric which is a Berwald metric. Let $(M,F)$ is a Finsler manifold.

- $(T_xM, F_x)$ : Minkowski space,
- $(T_xM, \hat{g}_x)$ : Rieman manifold. For any $y \in T_xM$, $y = (y^i)$, define $\hat{g}_x$ by $g_{ij}(x,y)\mathrm{d}y^i \otimes \mathrm{d}y^j$. For any $y \in T_xM$, $T_y(T_xM) \cong T_xM$.

**Proposition 17.1** (proposition 4.4.1, see [1]). *For any $c = c(t)$, $c(0) = p$, $c(1) = q$. Then $P_c\colon (T_pM, \hat{g}_p) \to (T_qM, \hat{g}_q)$ is isometric.*

*Remark* 17.1.
- $U \in M$ is open, and $C^\infty$ map $H : (-\varepsilon, \varepsilon) \times U \to M$
  - for any $x \in U$, $H(0,x) = x$. Here we set $H_t(x) = H(t,x)$.
  - for any $|s| < \varepsilon$, $|t| < \varepsilon$, and $|s+t| < \varepsilon$, $H_{s+t}(x) = H_s \circ H_t(x)$.

then we call $\{H_t\}$ is *one-parameter group of diffeomorphism acting $U$*.

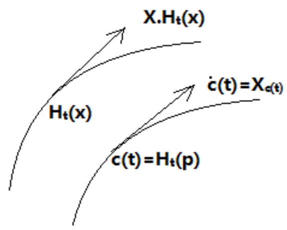



FIGURE 12. one-parameter group of diffeomorphism acting $U$

- Suppose that $X$ is a vector field on $M$. For any $p \in M$ there is a neighborhood $U \in M$, and a one parameter-diffeomorphism group $\{H_t\}$ such that $X|_{H_t(x)} = \frac{\mathrm{d}H_t(x)}{\mathrm{d}t}$, $x \in M$. Then $\{H_t\}$ is called the *flow of $X$*.

## Part 5. [15]Note of Chapter 5, S-Curvature

### 18. General Connections and The Identities[6]

(1)
- $F\Gamma = \left(F^i_{jk}, N^i_j, V^i_{jk}\right)$,
- $P^i_{jk}$-Landsberg Curvature.

(2) Ricci identities. It depict the symmetric of a tensor to the index.
 |- horizonal co-derivative, $\Big|$- vertical co-derivative.

(3) Bianchi identities. It is the relationship between kinds of curvatures and torsion and their derivatives, mainly derived from the structure equations.
 $(i,j,k)$ de note the residue part is circulate the $(i,j,k)$ in the first part.

[15]Ninth Time of Lecture, Thursday, May 6, 2010

(4) Chern connection. The sign in (4.4) contrary to the sign in our text book [1].
(5) Berwald connection. It is the connection which satisfy the (5.1) and (5.2).

## 19. Distortion(qibian in Chinese) and S-Curvature

19.1. **Distortion.** Recall that the Busemann-Hausdroff volume form is:

$$\sigma_F = \frac{\mathrm{vol}(B^n)}{\mathrm{vol}\{(y^i \in \mathbf{R^n} | F(y^i b_i < 1)\}}, \quad \mathrm{d}V_F = \sigma_F \theta^1 \wedge \theta^2 \cdots \wedge \theta^n.$$

in particular, if $F = \alpha$ is a Riemannian metric, then $\sigma_\alpha(x) = \sqrt{\det\left(g_{ij}(0)\right)}$.

Define

$$\tau(x, y) := \ln \frac{\sqrt{\det\left(g_{ij}(x, y)\right)}}{\sigma_F(x)},$$

we call it the *Distortion* of $F$, which measures the difference of Finsler metric and Riemannian metric. $\tau_{y^i} = g^{jk} C_{ijk} = \mathcal{I}_i$(mean Cartan curvature).

19.2. **S-Curvature.** For any $(x, y) \in TM$, there exist a geodesic $c = c(t)$ such that $c(0) = x$, $c'(0) = y$. Define the *S-curvature* as

$$S(x, y) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\tau(c(t), \dot{c}(t))\right]|_{t=0} = \tau_{|k}(x, y) y^k, \quad \text{(see [1], p76)}$$

P.S. $S$ denote Shen Zhong-ming, who have define S-curvature in Advance in math, 1997 by

$$S = \frac{\partial G^m}{\partial y^m}(x, y) - \frac{y^m}{\sigma_F} \frac{\partial \sigma_F}{\partial x^m}.$$

*Remark* 19.1. Let $\tilde{\sigma}_F(x)$ be another volume form of $F$. Put $\sigma_F(x) = \rho(x)\tilde{\sigma}_F(x)$. Then

$$\begin{aligned} S &= \frac{\partial G^m}{\partial y^m} - \frac{y^m}{\rho(x)\tilde{\sigma}_F}\left(\rho_{x^m}\tilde{\sigma}_F + \rho(x)\frac{\partial \tilde{\sigma}_F}{\partial x^m}\right) \\ &= \frac{\partial G^m}{\partial y^m} - \frac{y^m}{\rho(x)}\rho_{x^m} - \frac{y^m}{\tilde{\sigma}_F}\frac{\partial \tilde{\sigma}_F}{\partial x^m} \\ &= \tilde{S}(x, y) - (\mathrm{d}f)y. \end{aligned}$$

where $y = y^m \frac{\partial}{\partial x^m}$, $\mathrm{d}f = \frac{\rho_{x^m}}{\rho}\mathrm{d}x^m$. And let $\eta = -r\mathrm{d}f$, then $S(x, y) = \tilde{S}(s, y) + \eta(y)$. where $\eta$ is closed.

19.3. **Mean Berwald Curvature.** Suppose $(M, F)$ is a Finsler manifold, $G^i$ is the geodesic coefficient. $B_j{}^i{}_{kl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(x, y)$ is a linear transform on tangent space of order 3. We call the family $\left\{B_y | y \in TM_0\right\}$ is the *Berwald curvature*. A metric on $M$ is called a *Berwald metric* if $B = 0$, or equally, $B_j{}^i{}_{kl} = 0$, which equally to say $G^i_{jk} = G^i_{jk}(x)$.

By S-curvature, define a tensor of order 2 by

$$E_{ij}(x, y) = \frac{1}{2} S_{y^i y^j},$$

then $E = \left\{E_y | y \in TM_0\right\}$ is called the *mean Berwald curvature*. in fact, $S = \frac{\partial G^m}{\partial y^m} - \frac{y^m}{\sigma_F}\frac{\partial \sigma_F}{\partial x^m} \Rightarrow$ $E_{ij} = \frac{1}{2}S_{ij} = \frac{1}{2}\left[\frac{G^m}{\partial y^m}\right]_{y^i y^j} = \frac{1}{2}B_i{}^m{}_{jm}$.

19.4. **Berwald Spaces.**

**Proposition 19.1** (proposition 5.1.2, see [1]). *For any Berwald metric, the* S-curvature *vanishes, i.e.,* $S = 0$.

*Proof.* It should be noted that there we use the *Busemamn-Husdorff volume form*.

For any $(x, y) \in TM_0$, there exist a geodesic $\sigma = \sigma(t)$ such that $\sigma(0) = x, \sigma'(0) = y$. Let $\{e_i(t)\}$ is the parallel frame along $\sigma$, namely, at every point, the frame is get by a parallel transport. Then $g_{ij}(t) = g_{\dot{\sigma}(t)}(e_i(t), e_j(t)) \overset{4.1}{=\!=} \text{const}$.

On the other hand,

$$\tau = \ln\left(\frac{\sqrt{\det(g_{jk})}}{\sigma_F}\right), \quad \sigma_F = \frac{\mathrm{vol}(B^n)}{\mathrm{vol}\left\{(y^i) \in \mathbf{R^n} | F(x, y^i e_i < 1)\right\}},$$

an we have $F(\sigma(t), y^i e_i(t)) = \text{const}$. Note that the $y^i e_i(t)$ is a parallel vector field. Then we can get $\mathscr{U}_t : \left\{(y^i) \in \mathbf{R}^n | F(\sigma(t), y^i e_i(t) < 1)\right\}$, is independent of $t$. So $\mathrm{vol}\left(\Omega(\mathscr{U}_t)\right) = \text{const}$, which implies $\sigma_F$ is a const. □

19.5. **Isotropic, Almost Isotropic S-Curvature.**

(1)
- A Finsler metric $F$ on $M$ is of *isotropic S-curvature* if $S = (n+1)c(x)F$. where $n = \dim M$.
- A Finsler metirc $F$ on $M$ is *almost isotropic S-curvature* if $S = (n+1)\{cF + \eta\}$, where $c = c(x)$, and $\eta$ is a *closed 1-form*.

(2) A Finsler metric $F$ on $M$ is *isotropic mean Berwald curvature* if

$$E = \frac{1}{2}(n+1)cF^{-1}h, \quad \text{or equally, } E_{ij} = \frac{1}{2}(n+1)cF^{-1}h_{ij}.$$

where $h_{ij} = g_{ij} - l_i l_j$ is the *angular metric tensor*. $l_i = F_{y^i} = \frac{y_i}{F}$, and note that $h_{ij} = FF_{y^i y^j}$, so $E_{ij} = \frac{1}{2}(n+1)cF_{y^i y^j}$. Which is said that, $E_{ij}/h_{ij}$(act on two tangent vector fields) is constant along geodesic.

*Remark* 19.2. $S = (n+1)F \Rightarrow E + \frac{n+1}{2}cF^{-1}h$, conversely, we have to suppose $F$ is projectively flat. Left as a excise.

## 20. Randers metrics of Isotropic S-Curvature

20.1. Suppose $F = \alpha + \beta$ is a *Randers metric*. $G^i = G^i_\alpha + Py^i + Q^i$, $P := \frac{e_{00}}{eF} - s_0$, $Q^i = \alpha s^i$, $e_{00} = e_{ij}y^i y^j$, $e_{ij} = r_{ij} + b_i s_j + b_j s_i$.

On the other hand, $\mathrm{d}V_F = e^{(n+1)\rho(x)}\mathrm{d}V_\alpha$, so $e\sigma_F(x) = e^{(n+1)\rho(x)\alpha(x)}$, $\rho = \ln\sqrt{1 - \|\rho_x\|^2_\alpha}$.

*Remark* 20.1. Note of the formula above (5.10) see [1]: $G^i = \frac{1}{2}\Gamma^i{}_{jk}{}_{y^j y^k}$. On $\alpha$, $\frac{\partial G^m_\alpha}{\partial y^m} = \bar{\Gamma}^m_{im}y^i = y^m \frac{1}{2}a^{ml}\frac{\partial a_{ml}}{\partial x^i} = y^i \frac{\partial \ln\sqrt{\det(a_{ij})}}{\partial x^i} = \frac{y^m}{\sigma_\alpha}\frac{\partial \sigma_\alpha}{\partial x^m} \Rightarrow S_\alpha = \frac{G^m_\alpha}{\partial y^m} - \frac{y^m}{\sigma_\alpha}\frac{\partial \sigma_\alpha}{\partial x^m} = 0.$

Finally,

$$S = (n+1)\left\{\frac{e_{00}}{2F} - (s_0+\rho_0)\right\}$$

which is (5.10) on the text book.

**Lemma 20.1.** *$F = \alpha + \beta$, the following are equivalent,*

(*a*) $S = (n+1)cF$;
(*b*) $E = \frac{1}{2}(n+1)cF^{-1}h$;
(*c*) $e_{00} = 2c(\alpha^2 - \beta^2) \Leftrightarrow e_{ij} = 2c(a_{ij} - b_i b_j)$.

*Proof.*

- (a) ⇒ (b), is trivial.
- (b) ⇒ (c), $(b) \Rightarrow E_{ij} = \frac{1}{2}(n+1)cF^{-1}h_{ij} \Leftrightarrow S_{y^iy^j} = (n+1)cF_{y^iy^j}$. Then $(b) \Leftrightarrow S + (n+1)\{c(x)F + \eta\}$, $\eta$ is a 1-form. Integrate on both side, and note $S$ is homogenous of order 1 respect to $y$, by (5.10), $e_{00} = 2cF^2 + 2\theta F$, where $\theta = s_0 + \rho_0 + \eta$. So,

$$\boxed{\left\{e_{00} - 2c(\alpha^2+\beta^2) - 2\theta\beta\right\}} - \alpha\boxed{(4(\beta+2\theta))} = 0.$$

  As $\alpha$ is a irrational polynomial with respect to $y^i$, but the boxed part in the above formula is rational with respect to $y^i$. So we get

$$e_{00} - 2c(\alpha^2+\beta^2) - 2\theta\beta = 0, \quad 4(\beta + 2\theta) = 0.$$

  i.e.,

$$\theta = -2c\beta, \quad e_{00} = 2c(\alpha^2 - \beta^2).$$

- (c) ⇒ (a), By $e_{00} = 2c(\alpha^2-\beta^2)$ and (5.10). $S = (n+1)\{c(\alpha - \beta) - (s_0 + \rho_0)\}$. On the other hand, $\left(\text{note: } s_j = b^i s_{ij}, s_j b^j = b^i b^j s_{ij} = 0, \text{ by } s_{ij} = -s_{ji};\ b_j b^i = a^{ji} b_j b_i = \|\beta\|_\alpha^2\right)$.

$$\begin{aligned} b^j(e_{ij}) &= b^j(r_{ij} + b_i s_j + b_j s_i) = b^j r_{ij} + \|\beta\|_\alpha^2 s_i \\ &= 2c\left(b^j a_{ij} - b_i b_j b^j\right) \\ &= 2c(1 - \|\beta\|_\alpha^2) b_i. \end{aligned}$$

  $\rho = \ln\sqrt{1 - \|\beta\|_\alpha^2}$, $\|\beta\|_\alpha^2 = a^{ij}b_ib_j$, $\rho_i = \rho_{x^i} = \rho_{;i} = -\frac{a^{jk}b_{j;i}b_k}{1-\|\beta\|_\alpha^2}$, finally, $-b^j b_{j;i} = \left(1 - \|\beta\|_\alpha^2\right)\rho_i$. where $\rho_{;i}$ is the horizonal co-derivative with respect to $\alpha$. And $b^j b_{j;i} = b^j(r_{ji} + s_{ji}) = b^j r_{ij} + s_i$, $r_{ij}b^j = (1 - \|\beta\|_\alpha^2)\rho_i - s_i$. By which we have $(1 - \|\beta\|_\alpha^2)(\rho_i + s_i + 2cb_i) = 0$, note $\|\beta\|_\alpha^2 < 1$ so, $\rho_i + s_i + 2cb_i = 0$, or equally, $s_0 + \rho_0 = -2c\beta$. Thus, $S = (n+1)cF$.

□

## 21. [16]Non-Riemann Geometric Quantities

The Cartan tensor $C_{ijk}$, mean Cartan tensor $I_i$, and distortion $\tau$ are non-Riemann geometric quantities. That's to say, any of them vanish if and only if the metric is Riemannian.

We have the following diagram

$$\begin{array}{ccccc} C_{ijk} & \longrightarrow & \mathcal{I}_i = g^{jk}C_{ijk} = \tau_{y^i} & \longleftarrow & \tau \\ \downarrow & & \downarrow & & \downarrow \\ y^m\mathcal{B}^m_{ijk} = \mathcal{L}_{ijk} = C_{ijk|m}y^m & \longrightarrow & \mathcal{J}_i = \mathcal{I}_{i|m}y^m = g^{jk}\mathcal{L}_{ijk} & \longleftarrow & \mathcal{S} = \tau_{|m}y^m \end{array}$$

In the diagram, the following items are the rate of change of the above items along a geodesic respectively.

## 22.

Suppose $F = \alpha + \beta$, $S = (1+n)c(x)F \Leftrightarrow E_{ij} = \frac{n+1}{2}c(x)F^{-1}h_{ij} \Leftrightarrow e_{ij} = 2c(x)(a_{ij}(x) - b_i(x)b_j(x))$.

## 23. Navigation Data

*Navigation data* $(h, V) \Rightarrow$ Randers metric$F, h(x, y/F - V_x) = 1 \Leftrightarrow h(x, y - F(x,y)V_x) = F(x,y) \Leftrightarrow F(x,y) = \frac{\sqrt{\lambda h^2 + V_0^2}}{\lambda} - \frac{V_0}{\lambda}$, where $\lambda = 1 - \|V\|_h^2$ (see [1], P 54).

$$R_{ij}, \quad S_{ij}, \quad R_i, \quad s_i, \quad \cdots$$

$$\mathrm{d}V_F = \mathrm{d}V_h.$$

By lemma 9.1$\Rightarrow$ (5.13)$\cdots \Rightarrow$ (5.15[17]), which is the formula of S-curvature of a Randers metric.

[16] Tenth Time of Lecture, Thursday, May 13, 2010

[17] $S = \frac{n+1}{2F}\left\{2FR_0 - R_{00} - F^2R\right\}$

23.1. $F$ is of isotropic S-curvature. Put $\xi^i = y^i - F(x.y)V^i$, then $h(x, xi) = F(x, y)$, that is, $h_{ij}(x)\xi^i\xi^j = g_{ij}(x, y)y^iy^j$. By (5.15),

$$
\begin{aligned}
\frac{S}{F} &= \frac{n+1}{2F^2}\left\{2FR_{ij}V^iy^j - R_{ij}y^iy^j - F^2R_{ij}V^iV^j\right\} \\
&= \frac{n+1}{2F^2}\left\{\xi^iy^i - \xi^jy^j\right\}R_{ij} \\
&= -\frac{n+1}{2F^2}R_{ij}\xi^i\xi^j \\
&= -\frac{n+1}{2}\frac{R_{ij}\xi^i\xi^j}{h_{ij}\xi^i\xi^j} \qquad (Lemma 5.2.4)
\end{aligned}
$$

Note that there $R_{ij}, h_{ij}$ is only related to $x$.

**Proposition 23.1** (proposition 5.2.5, see [1]). *Let $F = \alpha + \beta$ given by navigation data $(h, V)$. Then $S = (n+1)c(x)F$ if and only if*

$$R_{ij} = -2c(x)h_{ij} \tag{23.1}$$

*the above equation is (5.18) in textbook*[1].

Use $\xi^i, \xi^j$ in Lemma 5.2.4 is arbitrary, we have $V_{i|j} + V_{j|i} = -4c(x)h_{ij}$. Where "|" is the horizonal derivatives with respect to $h$.

23.2. **The solution of** (23.1)**.** Here we just get some key steps in the method used in the textbook see [1].

- Lemma 5.2.6.

$$
\begin{aligned}
\bar{R}_i{}^m{}_{kj} + \bar{R}_j{}^m{}_{ki} + \bar{R}_k{}^m{}_{ji} &= \left(\boxed{\bar{R}_i{}^m{}_{kj} + \bar{R}_j{}^m{}_{ik} + \bar{R}_k{}^m{}_{ji}}\right)\boxed{-\bar{R}_j{}^m{}_{ik} + \bar{R}_j{}^m{}_{ki}} \\
&= 2\bar{R}_j{}^m{}_{ki}
\end{aligned}
$$

  In the first box, the item is equal to 0 and in the second box the item is skew-symmetry.
- Lemma 5.2.7. $(h, V)$, (5.16) for $c = c(x) \Rightarrow c$ and $V$ satisfy (5.24).

$$\bar{R}_{ijkl} = h_{jm}\bar{R}_i{}^m{}_{kl}.$$

$$V_{i|j|k|l} - V_{i|j|l|k} \overset{\text{view } V_{i|j} \text{ as co-vector of order 2.}}{\underset{\text{Ricci Identity}}{=\!=\!=\!=\!=}} V_{m|j}\bar{R}_i{}^m{}_{kl} + V_{i|m}\bar{R}_j{}^m{}_{kl}$$

  the Ricci Identity can cf.[6] P3, the first formula. $V_{i|j}$ is a function of $x$ and independent of $y$. Thus, the vertical derivatives vanish. By (23.1) $V_{i|m} + V_{m|i} = -4ch_{im}$. Form which we have

$$V_{i|m}\bar{R}_j{}^m{}_{kl} + V_{m|i}\bar{R}_i{}^m{}_{kl} = -4c\bar{R}_{jikl} = 4c\bar{R}_{ijkl}.$$

  from which we can derive (5.24).

- Lemma 5.2.8. $(h, V)$ satisfy (23.1). $h$ is of constant curvature $\mu$. Then $c$ satisfies

$$c_{|i|j} + \mu c h_{ij} = 0 \quad (n > 2),$$
$$\triangle c + 2\mu c = 0 \quad (n = 2).$$

or equally,

$$\triangle c + n\mu c = 0, \quad (n \geq 2).$$

*Proof.* $\bar{R}_k{}^m{}_{ij} = -\mu(h_{ki}h_{lj} - h_{kj}h_{li})$, by assumption,

$$\bar{R}_k{}^m{}_{ij}|l = 0, \quad \begin{pmatrix}\text{note the horizonal derivatives of } \mu \text{ vanish, and} \\ \text{the horizonal derivatives of } h \text{ vanish too.}\end{pmatrix}$$

$$V_{m|j}\bar{R}_i{}^m{}_{kl} = \mu(V_{k|j}h_{il} - V_{l|j}h_{ik}).$$

Plugging above formula into (5.24) we get (5.30).

If $n > 3$, we can chose $i \neq j$, $m \neq i, j$

$$c_{|i|i} + \mu c = -(c_{|j|j} + \mu c),$$
$$c_{|i|i} + \mu c = -(c_{|m|m} + \mu c),$$
$$\Rightarrow c_{|j|j} + \mu c = (c_{|m|m} + \mu c),$$

in particular,

$$c_{|i|i} + \mu c = -(c_{|i|i} + \mu c) \Rightarrow (5.32).$$

If $n = 2$, then by (5.30) we can get the conclusion.

You'd better try yourself to get (5.35) from (5.34). □

- Lemma 5.2.9 $c = c(x)$ satisfies (5.28) $\Rightarrow$

$$c = \frac{\lambda + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}}.$$

*Proof.* By (5.35) and $c_{|i} = c_x$, we get

$$c_{|i|j} = c_{x^i}c_{x^j} + \mu\frac{x^i c_{x^j} + x^j c_{x^i}}{1 + \mu|x|^2}.$$

Plugging $c_{|i|j}$ into (5.28) we get (try yourself) the idea: let $f = \sqrt{1 + \mu|x|^2}c$, then $f_{x^i x^j} = 0$. Form which we get $f_{x^i} = a_i$, $f = a_i x^i + \lambda = \langle a, x\rangle + \lambda \Rightarrow (5.36)$. □

**Proposition 23.2** (5.2.10, see [1]). *Let $h = h_\mu$ (constant curvature), $(h, V)$ Satisfies (23.1). $n \geq 3$ then $c$ is given by (5.36) and $V$ satisfies (5.37).*

*Remark* 23.1. • 2005, chen-shen get the result.

- 2002, $F = \alpha + \beta$, and with two conditions: projection flat (i.e., $\alpha$ is of constant curvature, $\beta$ is closed) and $S = (n+1)cF$, chen-shen have solved $c, F$.

*Proof.* case 1. $\mu = 0$, $c = \langle a, x\rangle + \lambda$, $h_{ij} = \delta_{ij}$ $V_{i|j} = \frac{\partial V_i}{\partial x^j}$.
case 2. $\mu \neq 0$, Let $P_i = V_i - \frac{2}{\mu} c_{|i} \Rightarrow P_{i|j} + P_{j|i} = 0$. □

## 24. An Equation on the S-Curvature

The equation of (5.42) is quite important, for example, the classification of Randers metric of projective flat and of isotropic S-curvature. Or the classification of Randers metric of scalar flag curvature and of isotropic S-curvature.

**Lemma 24.1** (Lemma 5.3.1)**.**

$$S_{\cdot k|m} y^m - S_{|k} = \mathcal{I}_{k|p|q} y^p y^q + \mathcal{I}_m R^m_k \tag{5.41}$$

$$S_{\cdot k|m} y^m - S_{|k} = -\frac{1}{3}\left\{2R^m_{k\cdot m} + R^m_{m\cdot k}\right\} \tag{5.42}$$

(24.1)

*Proof.* $\mathrm{d}\tau = \tau_{|l}\omega^l + \tau_{\cdot i}\omega^{n+i}$ by (5.43) and the definition of derivatives. $0 = \mathrm{d}^2\tau = \mathrm{d}\tau_{|l} \wedge \omega^l + \tau_{|l}\mathrm{d}\omega^l + \mathrm{d}\mathcal{I}_i \wedge \omega^{n+i} + \mathcal{I}_i \mathrm{d}\omega^{n+i}$. Note $\mathrm{d}\tau_{|l} - \tau_{|m}\omega^m_l$ is the derivative. So, $\mathrm{d}\tau_{|l} = \tau_{|m}\omega^l_m + \tau_{l|m}\omega^m + \tau_{l\cdot l}\omega^{n+m}$, $\mathrm{d}\mathcal{I}_i = \mathcal{I}_l\omega^l_i + \mathcal{I}_{i|l}\omega^l + \mathcal{I}_{i\cdot l}\omega^{n+l}$, $\mathrm{d}\omega^{n+i} = \Omega^i - \cdots$, by the decompose formula of $\Omega$, we get the formula above (5.45) and finally we have (5.45) and then $\mathcal{I}_{l|k} = \tau_{\cdot l|k}$. □

## Part 6. Chapter 6. Riemann Curvature

## 25. Riemann Curvature

25.1. **Definition.** Suppose $(M, F)$ is a Finsler manifold, then the *Riemannian curvature tensor* $R^i_k(x,y)$ see (2.43).

For any $y \in T_xM \setminus \{0\}$; Define $R_y : T_xM \to T_xM$ by $T_xM \ni u \mapsto R_y(u) = R^i_k(x,y)u^k e_i$, i.e., $R_y = R^i_k(x,y)e_i \otimes \theta^k$, where $\theta$ is the dual basis of $e$. The map $R_y$ is a linear transform on $T_xM$.

Write $R := \left\{R_y | y \in T_{M_0}\right\}$, which is the *Riemannian curvature*.

*Remark* 25.1.
- $R_y(y) = 0$, $(R^i_k(x,y)y^k = 0)$;
- $R_{\lambda y} = \lambda^2 R_y$, for any $\lambda > 0$;
- $R_{ik}(x,y) =_{ki} (x,y)$; $R_{ik} = g_{im}R^m_k \Rightarrow g_y(R_y(u), v) = g_y(u, R_y(v))$, i.e., $R_y$ is self-adjoin of $g$. Further, For any $y \in T_xM$, $P = \mathrm{span}\{\mathrm{y},\mathrm{u}\} \subset \mathrm{T_xM}$. Note that the identity of $y$ and $u$ is quite different.

Define

$$\mathcal{K}(p,y) = \frac{g_y(R_y(u),u)}{g_y(y,y)g_y(u,u) - \left(g_y(y,u)\right)^2} = \frac{R_{ij}u^iu^j}{\left(F^2(x,y)g_{ij} - g_{ik}g_{jl}y^ky^l\right)u^iu^j}.$$

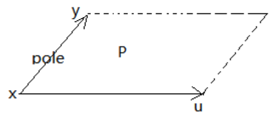



FIGURE 13. The flag on the tangent space of a metric at a point

which is the *flag curvature*.

*Remark* 25.2. $h_{ij} = g_{ij} - l_i l_j$, where $l_i = y_i/F$. $h_{ij}$ is the *angular metric tensor*.

Let $\mathbf{Ric}(x,y) = \sum_{i=1}^n g^{ij} g_y(R_y(e_i), e_j)$ is called the *Ricci curvature*. $g_y(R_y(e_i), e_j) = g_y(R_i^k e_k, e_j) = R_i^k g_{kj} = R_{ij}$. $g^{ij} g_y(R_y(e_i), e_j) = g^{ij} \boxed{R_{ij} = R_{ji}} = R_i^i$(= $\mathrm{trR_y}$ is the mean value of Riemannian curvature.)

*Remark* 25.3. Here **Ric** curvature is in the direction $y$.

**Proposition 25.1** (proposition 6.1.1, see [1]). *Suppose* $(M, F)$ *is a Riemannian manifold, then* $\mathcal{K}(p, y) = \mathcal{K}(p)$.

*Proof.* $R_k^i(x,y) = R_j{}^i{}_{kl}(x,y) y^j y^l \Rightarrow R_{ik}(x,y) = R_{jikl} y^j y^l$. Note they are only depended on $x$ when it is Riemannian metric. □

**Definition 25.1.**
- Suppose $F$ is of *scalar flag curvature*, i.e., $\mathcal{K}(p,y) = \mathcal{K}(x,y)$ it means independent of $p$, the curvature is isotropic on the tangent bundle (only relate to $(x, y)$, is the point of the underline manifold, independent of the tangent vector of $TM$). Note here the definition is not the same as the definition of scalar flag curvature in Riemann geometry, in which, is defined by the sum of Ricci curvature in every direction of a orthogonal basis. We are not define a scalar flag curvature in the sense of Finsler in Riemannian geometry.
- $F$ is *Einstein metric* with *Einstein scalar function* $\mu(x)$ if and only if $\mathbf{Ric} = (n-1)\mu(x)F^2$.

*Remark* 25.4.
- If $F$ is Riemannian metric. When $n > 3$, $F$ is Einstein metric, then $\mu(x)$ is a constant function.
- When $n = 3$, $F$ is Einstein metric, then $F$ is of constant curvature $\mu$.

In 2003, Bao-Robles proved that for Randers metric $F = \alpha + \beta$ the conclusion hold too.

*Remark* 25.5. By (6.3) $F$ of scalar curvature ($\mathcal{K}(p, y)$ is independent of $p$) if and only if $R_k^i = (n+1)\mathcal{K}(x,y) h_k^i$. Put $\mathbf{Ric}_{ij} := \left(\frac{1}{2}\mathbf{Ric}\right)_{y^i y^j}$ is Einstein if and only if $\mathbf{Ric}_{ij} = (n-1)\mu(x) g_{ij}$ this is the same as the definition of Einstein in Riemannian geometry.

25.2. [18]

**Proposition 25.2** (proposition 6.1.3, see [1]). *Any projectively flat Finsler metric is of scalar flag curvature.*

*Proof.* Suppose $(M, F)$ is a Finsler manifold, and projectively flat, then $G^i = Py^i$, where $P$ is a positively homogeneous function of degree 1 with respect to $y$, and being called the *projective factor*. In fact, $P = \frac{F_{x^k}y^k}{2F}$. By (2.49) (see [1]) $R^i_k = \Xi\delta^i_k + \tau_k y^i$, where $\Xi := P^2 - P_{x^k}y^k$, $\tau_k := 3(P_{x^k} - PP_{y^k}) + \boxed{\Xi_{y^k} = \Xi_{\cdot k}} \Rightarrow \tau_k y^k = -\Xi$.

$R_{jk} = g_{ij}R^i_k = g_{jk}\Xi + \tau_k y_j$, note $R_{jk} = R_{kj}$, then $\tau_k g_{ij}y^i = \tau_j g_{ik}y^i$, equally, $\tau_k g_j = \tau_j g_k \Rightarrow \tau_k F^2 = -\Xi y_k \Rightarrow \tau_k = -\frac{\Xi}{F^2}y_k \Rightarrow R^i_k = \Xi(\delta^i_k - l^i l_k) = \frac{\Xi}{F^2}F^2 h^i_k \Rightarrow \mathcal{K} = \frac{\Xi}{F^2} = \frac{P^2 - P_{x^k}y^k}{F^2} \Rightarrow (R^i_k = \mathcal{K}(x,y)F^2 h^i_k,\ h^i_k = \delta^i_k - l^i l_k \Rightarrow$ ofscalarflagcurvature). □

*Remark* 25.6.
- A Riemannian metric is projectively flat if and only if it is of constant curvature (Bertrand theorem);
- In Finsler geometry, is not the case, D.Bao and Z.Shen give a counter example use some method of Lie group in 2001, showed that a Finsler metric of constant (?flag) curvature may not projectively flat.

*Example* 25.1 (example 6.1.4). Compute yourself.

*Example* 25.2 (example 6.1.5). 2001, Z.Shen construct it, is projectively flat.

$$\text{Randers metric of constant flag curvature}\begin{cases}\text{Locally Minkowski metric}\\ \text{isometric to (6.9) under a conformal map}\end{cases}$$

25.3. **Riemann Curvature of Randers Metrics.** Assume that $\bar{F}$ and $F$ are projectively related, $\bar{G}^i = G^i + Py^i$, $P = \frac{\bar{F}_{;k}y^k}{2\bar{F}} \Rightarrow \bar{R}^i_k = R^i_k + \Xi\delta^i_k + \tau_k y^i$. where $\Xi := P^2 - P_{;k}y^k$, ";" is the horizonal derivative with respect to $\bar{F}$. $\tau_k := 3(P_{;k} - PP_{\cdot k}) + \Xi_{\cdot k}$, $\tau_k y^k = -\Xi$. Then, $\overline{\mathbf{Ric}} = \mathbf{Ric} + (n-1)\Xi$.

Randers metric $F = \alpha + \beta$, $\beta$ is Closed. $G^i = G^i_\alpha + Py^i + Q^i$, $P = \frac{e_{00}}{2F} - s_0$, $Q^i = \alpha s^i_0$. By assume, $\beta$ is closed, $s_{ij} = 0 \Rightarrow s_i = 0$ and $F, \alpha$ is projectively related, so

$$G^i = G^i_\alpha P y^i, \quad P = \frac{r_{00}}{2F}, \quad r_{ij} = \frac{1}{2}(b_{i;k} + b_{j;i}), \quad r_{00} = y^i y^j r_{ij} \Rightarrow P = \frac{b_{i;j}y^i y^j}{2F}.$$

Put $\Phi = B_{i;j;k}y^i y^j y^k$. Then by $b_{i;j=b_{j;i}} \Leftrightarrow s_{ij} = 0$ we'll get (6.10).

Next is an important example.

*Example* 25.3 (Example 6.1.6, see [1]). Let $(S^n, \alpha) \subset R^{n+1}$ is a standard unit sphere, and $\alpha$ is the standard metric (i.e., set $\mu$ in example 6.1.4 by 1).

$$\begin{array}{ccccc} R^n & \xrightarrow{project} & S^n & \xrightarrow{inclusion} & R^{n+1} \\ ? & \longrightarrow & (u^a) & \longrightarrow & (x^i), \end{array}$$

[18]Eleventh Time of Lecture, Thursday, May 20, 2010

and define $f$ by

$$f\colon S^n \to R$$
$$x \mapsto \varepsilon x^i,$$

where the $i$ is fixed. Then, $\alpha = \alpha_{+1}$, $a_{ij} = \frac{\delta_{ij}}{1+|x|^2} - \frac{x^i x^j}{(1+|x|^2)^2}$, $\bar{\Gamma}^i_{jk} = -\frac{x_j\delta^i_k + x^k\delta^i_j}{1+|x|^2} \Rightarrow$ (6.12), $f_{;i;j} = -\delta_{ij} f$.

Define $\delta := f^2(x) + |\mathrm{d}f_x|^2$, then it must be constant. In fact, $\delta_{x^k} = \delta_{;k} = 2ff_{;k} + (a_{jl} f_{x^j} f_{x^l})_{;k} = 2ff_{;k} + 2\delta^{jl} f_{;j;k} f_{;l} = 0$.

Define $F = \alpha + \beta$, $\beta = b_i y^i$, $b_i(x) = -\frac{f_{;i}(x)}{\sqrt{1-f^2}}$, then $\beta$ is closed,. and

- the great circles have F-length of $2\pi$;
- compute the S-curvature;
- compute the flag-Curvature.

In the computation of flag-curvature, use the Riemmann curvature, $\alpha$ is of constant curvature and $\beta$ is closed we have $F$ is projectively flat. $\mathbf{Ric} = (n+1)\mathcal{K}(x,y)F^2 \Rightarrow \mathcal{K}(x,y) = \frac{\mathbf{Ric}}{(n-1)F^2}$. and by (6.10) to get the **Ric** curvature.

*Example* 25.4 (example 6.1.7, Hilbert metric on the strong convex domain.). $H(x,y) = \frac{1}{2}\left\{\theta(x,y) + \bar{\theta}(x,y)\right\}$, note that $\theta_{x^k} = \theta\theta_{y^k} = \frac{1}{2}\left\{\theta^2\right\}_{y^k}$, $\bar{\theta}(x,y) = \theta(x,-y)$. Then, $H$ is projectively flat $\Rightarrow G^i = Py^i$, $P = \frac{1}{2}(\theta - \bar{\theta}) \Rightarrow \mathcal{K}(x,y) = \frac{P^2 - P_{x^k} y^k}{H^2} = -1$.

## 26. Second variation of a Geodesic

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- by the first variation: preserving the end points, get the differential equation of geodesic.
- by the second variation: preserving the geodesic, get the variation field is Jacobi field, and induce the Riemann curvature.

26.1. **preserving the end points.** $(M,F)$ is Finsler manifold, $\sigma = \sigma(t)$, $a \le t \le b$ is a geodesic. Let $H : [a,b] \times (-\varepsilon,\varepsilon) \to M$ is a $C^\infty$ map, with $H(a,s) = \sigma(a)$, $H(m,s) = \sigma(b)$. Set $\sigma_s := \sigma_s(t) := H(t,s)$, $t \in [a,b]$, $|s| < \varepsilon$. Let $V(t) := \frac{\partial H}{\partial s}(t,0)$ is the variation field. see figure 14. Let $\mathcal{L}(s) = \int_a^b F(\sigma_s, \dot{\sigma}_s)\mathrm{d}t = \int_a^b F(H(t,s), \frac{\partial H}{\partial t}(t,s))\mathrm{d}t = \int_a^b \left\{g\left(\frac{\partial H}{\partial t}(t,s), \frac{\partial H}{\partial t}(t,s)\right)\right\}^{\frac{1}{2}} \mathrm{d}t$, $g = g_{\frac{\partial H}{\partial t}}$. $\Rightarrow \mathcal{L}'(0) = 0$.

$$\left.\begin{aligned} V^\top(t) &= g_{\dot{\sigma}}(V(t), \dot{\sigma}(t))\dot{\sigma}(t), \\ V^\perp(t) &= V(t) - V^\top(t). \end{aligned}\right\} \Rightarrow \mathcal{L}'(0) = \cdots, \mathcal{L}''(0) = \cdots \Rightarrow (6.15)$$

*Remark* 26.1. Note that the formula expressed by (6.15) is not easy at all, you can first review the correspond case in Riemann geometry, and the reference book is Cheeger & Ebin's the Comparison theorems in Riemannian geometry, see [9].

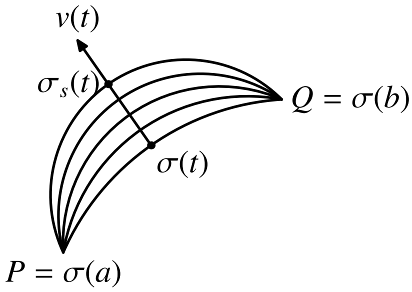
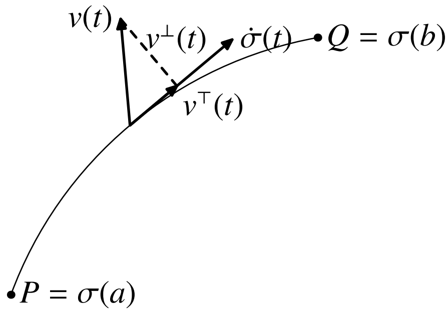



FIGURE 14. The Variation Field of First Variation




26.2. **perverting geodesic**[19]**.** Let $\sigma = \sigma(t)$, $a \le t \le b$ is a geodesic. $H : (a, b)\times(-\varepsilon, \varepsilon) \to M$ is $C^\infty$ such that $\sigma_s = \sigma_s(t) := H(t, s)$ is geodesic.

Put $J(t) = \frac{\partial H}{\partial s}(t, 0)$ is the *variation vector field*.

Put $T = \frac{\partial H}{\partial t}(t, s) = T^i(t, s)\frac{\partial}{\partial x^i}$, $T^i = \frac{\partial H^i}{\partial t}$. $U = \frac{\partial H}{\partial s}(t, s) = U^i(t, s)\frac{\partial}{\partial x^i}$, $U^i = \frac{\partial H^i}{\partial s}$. By $\frac{\partial^2 H^i}{\partial t^2} +$

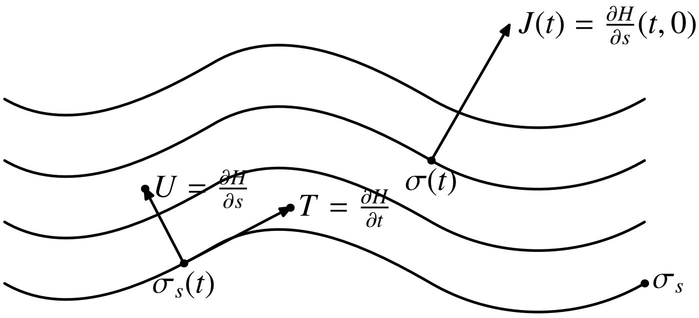



FIGURE 15. Second Variation Vector Field and Jacobi Field

[19] Twelfth Time of Lecture, Thursday, May 27, 2010

$2G^i(H, \frac{\partial H}{\partial t} = 0 \Rightarrow \frac{\partial T^i}{\partial t} + 2G^i(H,T) = 0$ is the equation (6.18) then note

$$\frac{\partial T^i}{\partial s} = \frac{\partial^2 H^i}{\partial s \partial t} = \frac{\partial U^i}{\partial t},$$

$$\frac{\partial}{\partial s}\left[G^i(H,T)\right] = U^k \frac{\partial G^i}{\partial x^k}(H,T) + \boxed{\frac{\partial T^k}{\partial s}\frac{\partial G^i}{\partial y^k}(H,T) = \frac{\partial U^k}{t} N^i_j(H,T)},$$

$$\frac{\partial}{\partial t}\left[N^i_j(H,T)\right] = T^k \frac{\partial N^i_j}{\partial x^k}(H,T) + \frac{\partial T^k}{\partial t}\frac{\partial N^i_j}{\partial y^k}(H,T)$$

$$= T^k \frac{\partial N^i_j}{\partial x^k}(H,T) - 2G^k(H,T)\frac{\partial N^i_j}{\partial y^k}(H,T). \qquad (6{,}19)$$

differential (6.18) with respect to $s$: $\frac{\partial}{\partial s}\left(\frac{\partial T^i}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial T^i}{\partial s}\right) = \frac{\partial^2 u^i}{\partial t^2} = -2U^k \frac{\partial G^i}{\partial x^k}(H,T) - 2\frac{\partial U^j}{\partial t} N^i_j(H,T)$. Then, recall that,

$$D_T U = \left(\frac{\partial U^i}{\partial t} + U^j N^i_j(H,T)\right)\frac{\partial}{\partial x^i}.$$

Use twitch of this we'll get

$$D_T D_T U = D_T\left(\left(\frac{\partial U^i}{\partial t} + U^j N^i_j(H,T)\right)\frac{\partial}{\partial x^i}\right)$$

$$((6.20)) \qquad = -U\left\{2\frac{\partial G^i}{\partial x^k} - T^j \frac{\partial N^i_k}{\partial x^j} + 2G^j \frac{N^i_j}{\partial y^k} - N^i_j n^j_k\right\}\frac{\partial}{\partial x^i}.$$

So, then by (6.20) we can express the Riemann tensor $R^i_k$ as

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - T^j \frac{\partial N^i_k}{\partial x^j} + 2G^j \frac{N^i_j}{\partial y^k} - N^i_j n^j_k \frac{\partial}{\partial x^i}.$$

$\Rightarrow D_T D_T U + \mathcal{R}_T(U) = 0$, taking $s = 0 \Rightarrow D_{\dot\sigma} D_{\dot\sigma} J + \mathcal{R}_{\dot\sigma}(J) = 0$.

*Example* 26.1 (The existence of the variation preserving the geodesic). For any geodesic $\sigma = \sigma(t)$, $a \le t \le b$, satisfying $\sigma(0) = x$, $\dot\sigma(0) = y$. Define $H : (a,b) \times (-\varepsilon, \varepsilon) \to M$ by $H(t,s) = \exp_x(t(y + sv))$, such that $\sigma_s(t) := H(t,s)$ is geodesic. $J(t) := \frac{\partial H}{\partial s}(t,0)$[20]c.f. Cheeger & Ebin. [9] $(\exp_x)_*\big|_{ty}(tv)$

## 26.3. **Sphere theorem.**

- Riemann geometry: If a manifold $M$ is
  (1) simply connected;
  (2) closed;
  (3) $1/4 < \mathcal{K}$(section curvature)$\le 1$.
  then $M$ is homeomorphic to the n-sphere. see [8].

[20] =

- Finsler geometry: $(M,F)$ is a Finsler manifold, Let $\lambda = \sup_{(x,y)\in TM_0} \frac{F(x,-y)}{F(x,y)}$, then, $\lambda \geq \frac{F(x,-y)}{F(x,y)}$, for any $(x,y) \in TM_0$. and so $\lambda \geq \frac{F(x,-y)}{F(x,y)} \geq \frac{1}{\lambda} \Rightarrow \lambda \geq 1$, and with equality hold if and only if $F(x,y) = F(x,-y)$, i.e., $F$ is reversible.

**Theorem 26.1** (theorem 6.2.1)**.** *$(M,F)$ is a Finsler manifold, and*

*(1) Simply connected;*
*(2) closed;*
*(3) $(1 - \frac{1}{1+\lambda})^2 < \mathcal{K} \leq 1$ $(n \geq 3)$.*

*then $M$ is homotopy to $n$-sphere.*

## 27. Non-positive flag curvature

Recall that $P = \operatorname{span}\{y, v\}$, $\mathcal{K}(P,y) = \frac{g_y(R_y(v),v)}{g_y(y,y)g_y(v,v) - g_y^2(y,v)}$.

27.1. $(M,F)$ is a Finsler manifold. Then there exist $H : (a,b) \times (-\varepsilon,\varepsilon) \to M$ by $H(t,s) = \exp_x(t(y+sv))$, $J(t) = (\exp_x)\big|_{ty}(tv) \Rightarrow J(0) = 0, (D_{\dot\sigma}J)(0) = v$. Conversely, if a Jacobi field $J = J(t)$ satisfying (6.23) then there is $\cdots$ *conjugate points*: is the farest point that

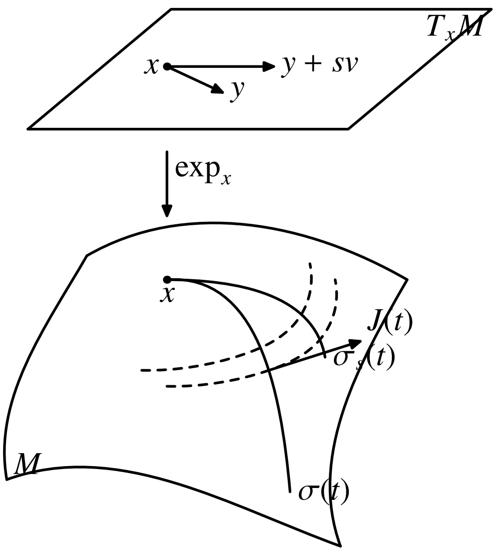



Figure 16. Second Variation Vector Field and Jacobi Field

preserving the property of the shortest path between any two points is a geodesic.

**Theorem 27.1.** *$(M,F)$ is a Finsler manifold, with*

*(1) positively complete;*
*(2) $\mathcal{K} \leq 0$;*

*then, $\exp_x : T_xM \to M$ is non-singular.*

*Proof.* For any $(x,y) \in TM_0$, there exists one and only one geodesic $\sigma$, such that $\sigma(t) = \exp_x(ty)$, $\sigma(0) = x$, $\dot\sigma(0) = y$. Assume that $J(t)$ is the Jacobi Field, satisfying $J(0) = 0$, $D_{\dot\sigma}J = v$. Let $f(t) := g_{\dot\sigma(t)}(J(t), J(t))$, $t \geq 0$.

Recall that: chern connection compatible with the metric $\Leftrightarrow$ $\mathrm{d}g_{ij} - g_{kj}\omega_j^k - g_{ik}\omega_j^k = 2C_{ijk}\omega^{n+k} \Leftrightarrow W(g_y(U,V)) = g_y(\Delta_W^y U, V) + g_y(U, \Delta_W^y V) + 2C_{\dot{y}}(U, V, \Delta_W^y y)$.

Now we have,

$$\begin{aligned} f'(t) &= \Delta_{\dot\sigma}(g_{\dot\sigma(t)}(J(t), J(t))) \\ &= g_{\dot\sigma}\left(\Delta_{\dot\sigma}^{\dot\sigma} J(t), J(t)\right) + g_{\dot\sigma}\left(J(t), \Delta_{\dot\sigma}^{\dot\sigma} J(t)\right) \\ &\quad + \boxed{2C_{\dot\sigma}\left(J(t), J(t), \Delta_{\dot\sigma}^{\dot\sigma}\dot\sigma\right) = 0} \\ &= 2g_{\dot\sigma}\left(\Delta_{\dot\sigma} J(t), J(t)\right). \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}f'(t) &= \int_0^t \boxed{\frac{\mathrm{d}}{\mathrm{d}t} = \Delta_{\dot\sigma}} (g_{\dot\sigma}(\Delta_{\dot\sigma} J(t), J(t)))\,\mathrm{d}t. \\ &= \int_0^t (g_{\dot\sigma}(\Delta_\sigma \Delta_{\dot\sigma} J, J) + g_{\dot\sigma}(\Delta_{\dot\sigma} J, \Delta_{\dot\sigma}) + 0)\,\mathrm{d}t \\ &= \int_0^t (g_{\dot\sigma}(\Delta_{\dot\sigma} J, \Delta_{\dot\sigma} J) - g_{\dot\sigma}(R_{\dot\sigma}(J), J))\,\mathrm{d}t, \\ \frac{1}{2}f''(t) &= g_{\dot\sigma}(\Delta_{\dot\sigma} J, \Delta_{\dot\sigma} J) - g_{\dot\sigma}(R_{\dot\sigma}(J), J). \end{aligned}$$

By $\mathcal{K} \leq 0$, then $f'(t) \geq 0$, $f''(t) \geq 0$, and $\frac{1}{2}f''(0) = g_{\dot\sigma}(V, V) > 0$, which means $f(t)$ is none decrease, and convex. Finally, $f(t) > 0$, for any $t > 0$. □

27.2. Assume that $F(x, -y) = F(x, y)$ (reversible), then

$$g_{ij}(x,y) = \frac{1}{2}\left[F^2(x,y)\right]_{y^i y^j} = \frac{1}{2}\left[F^2(x,-y)\right]_{y^i y^j} = g_{ij}(x,-y).$$

thus,

$$\tau(x,y) = \ln\left(\frac{\sqrt{\det g_{ij}(x,y)}}{\sigma_F}\right) = \ln\left(\frac{\sqrt{\det g_{ij}(x,-y)}}{\sigma_F}\right) = \tau(x,-y).$$

Further more, $S(x,y) = \tau_{|m}(x,y)y^m = -\tau_{|m}(x,-y)(-y^m) = -S(x,-y)$. So, $F$ is of isotropic S-curvature if and only if $S(x,y) = (n+1)c(x)F(x,y) = (n+1)c(x)F(x,-y)$, on the other hand, by $S(x,y) = -S(x,-y) = -(n+1)c(x)F(x,y) = -S(x,y) \Rightarrow S(x,y) = 0$.

**Theorem 27.2** (theorem 6.3.2). *Suppose $(M, F)$ is a Finsler manifold, with the properties stated below:*

- *complete;*
- *$\|I\| \leq I_0 < +\infty$;*
- *$\mathcal{K} \leq 0$;*
- *$S = (n+1)cF$, is constant.*

*then*

- *$J = 0$, $R_y(I_y) = 0$;*

- *more over, if $\mathcal{K} < 0$, then $F$ is a Riemannian.*

*Proof.* Taking basis $\{b_i\} \overset{\text{dual}}{\longleftrightarrow} \left\{\theta^i\right\}$. Let $I_y = I^i b_i$, $I^i = g^{ij} I_j$ (Lagrange transformation) and $J_y = J^i b_i$, $J^i = g^{ij} J_j$. we have $R_y(I_y) = R^i_m I^m b_i$.
**step 1.** $S = (n+1)cF$, $c =$ const. note $F_{|k} = 0, F_{\cdot k|m} - F_{|m\cdot k} = 0$. so $S_{\cdot k|m} y^m - S_{|k} = 0 \Rightarrow J_{k|m} y^m + I_m R^m_k = 0$, see p.104. Contract by $g^{ik}$, $\Rightarrow J^i_{|m} y^m + R^i_m I^m = 0$, which is (6.24).
**step 2.** for any $(x, y) \in TM_0$, there exist one and only one geodesic $\sigma = \sigma(t)$ such that $\sigma(0) = x$, $\dot\sigma(0) = y$ by the completeness of $F$. Here $\sigma = \sigma(t)$ is defined on $R$, i.e. $-\infty < t + \infty$.

Let $I(t) := I^k(\sigma(t), \dot\sigma(t)) b_{k|\sigma(t)}$, $J(t) := J^k(\sigma(t), \dot\sigma(t)) b_{k|\sigma(t)}$, $J_i = I_{i|m} y^m \Rightarrow J^i + I^i_{|m} y^m$. $\Delta_{\dot\sigma} I(t) = \dot\sigma^P(t) I^k_{|p}(\sigma, \dot\sigma) b_{k|\sigma}$, note $I^k_{|m} = \frac{\delta I^k}{\delta x^p} + I^m \Gamma^k_{mp}$, then $\Delta_{\dot\sigma} I(t) = b_k \left(\mathrm{d}I^k(\dot\sigma) + I^m(t) \Gamma^k_{mp} \omega^l(\sigma)\right)$. Analogously, $\Delta_{\dot\sigma} J(t) = J^k_{|p}(\sigma, \dot\sigma) \dot\sigma^p b_{k|\sigma(t)} = \Delta_{\dot\sigma}\Delta_{\dot\sigma} I(t) \Rightarrow \Delta_{\dot\sigma}\Delta_{\dot\sigma} I(t) + R_{\dot\sigma}(I(t)) = 0 \Leftrightarrow \Delta_{\dot\sigma} J(t) + R_{\dot\sigma}(I(t)) = 0$. So, the mean Cartan torsion is a Jacobi field along any geodesic.
**step 3.** Let $\phi(t) := g_{\dot\sigma(t)(I(t), I(t))}$ (use the same method as theorem 6.3.1), we can reach (6.28).

Now note by assumption $\mathcal{K} \le 0 \Rightarrow \phi'' \ge 0 \Rightarrow \phi'(t) \ge 0$. Suppose that $\phi'(t_0) \ne 0$, if $\phi'(t_0 < 0) \Rightarrow \phi(t) = \phi(t_0) + \phi'(\xi)(t - t_0), \xi \in (t, t_0)$. $\phi'(\xi) \le \phi'(t) < 0 \Rightarrow \phi(t) \ge \phi(t_0) - \phi'(t_0)(t_0 - t)$. If $\phi'(t_0) > 0 \cdots \Rightarrow \phi(t) \ge \phi(t_0) + \phi'(t_0)(t - t_0), t > t_0$. In each case, we have $\lim_{t\to\infty} \phi(t) = +\infty$. This is a contradiction to (6.28). So, we must have $\phi'(t) = 0 \Rightarrow \phi''(t) = 0$ by (6.28). And $J(t) = 0 \Rightarrow R_{\dot\sigma(t)}(I(t)) = 0$.
**step 4.** If $\mathcal{K} < 0$, $g_y(I_y, y) = g_{ij}(x, y) I^i y^j = I_j y^j = 0 \Rightarrow I^y \perp y$. Now if $I_y \ne 0$, taking $P = \operatorname{span}\left\{I_y, y\right\}$, $\mathcal{K}(P, y) = \frac{R_y(R_y(I_y, I_y))}{g_{ij}(I_y, I_y) g_{ij}(R_y(I_y), R_y(I_y)) - g^2_{ij}(R_y(I_y), I_y)} = 0$ is a contradiction, so $I_y = 0$ and by *Deicke's theorem* $F$ is a Riemannian manifold. □

**Corollary 27.1.** *$(M, F)$ is a Finsler manifold with the following properties:*

- *Closed;*
- $S = (n+1)cF$, $c =$ const.*;*
- $\mathcal{K}(P, y) < 0$. *(note: closed $\Rightarrow$ complete and $\|I\| \le I_0 < +\infty \Rightarrow F$ is Riemannian.)*

*Remark* 27.1. Mo-Shen (2003) $(M, F)$ is Finsler manifold, dim $\ge 3$. and

- closed;
- $\mathcal{K} = \mathcal{K}(x, y)$, i.e. of scalar curvature;
- $\mathcal{K} \le 0$.

then, $F$ is a Randers metric.

*Example* 27.1 (Fish tank model). $h = |\cdot|$- Euclidean. $W = (-y, x, \bar P) \in T_P\mathbf{R}^n$. It's an navigation problem.

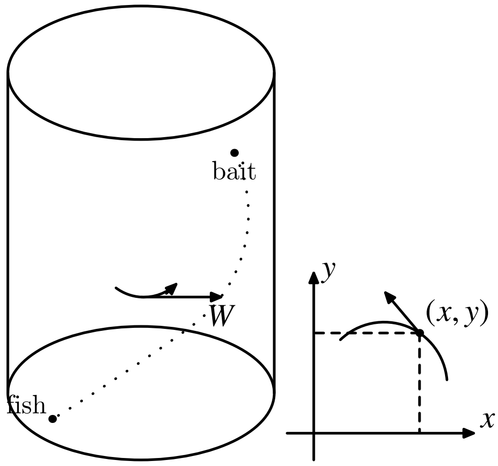



FIGURE 17. Fish Tank model

## Part 7. Finsler Metric of Scalar Curvature[21]

Recall that, $\mathcal{K}(P,y) = \frac{g_y(R_y(U),U)}{g_y(y,y)g_y(U,U)-g_y^2(y,U)}$, $P = \text{span}\{y, U\} \subset T_xM$.

### 28. BASIC PROPERTIES

28.1. $(M, F)$ is of scalar flag curvature $\Leftrightarrow R^n_k = \mathcal{K}F^2h^i_k = \mathcal{K}F^2\left(\delta^k_i - l_k l^i\right)$, $l_k = F_{y^k}$. Further, $R^i_{kl} = \frac{1}{3}\left(R^i_{k.l} - R^i_{l.k}\right) \Rightarrow R^i_{kl} = \frac{1}{3}\mathcal{K}_{.l}F^2h^i_k - \frac{1}{3}\mathcal{K}_{.k}F^2h^i_l + \mathcal{K}\left\{g_{lp}\delta^i_k - g_{kp}\delta^i_l\right\}y^p$, which is equation (7.2).

See the content of [3] page.40, Lemma 2.4.1, we have (7.4) and (7.5) as bellow:

$$C_{ijk|p|q}y^py^q = \frac{1}{3}F^2\left\{\mathcal{K}_{.i}h_{jk} + \mathcal{K}_{.j}h_{ik} + \mathcal{K}_{.k}h_{ij} + 3\mathcal{K}C_{ijk}\right\},$$

$$\mathcal{I}_{k|p|q}y^py^q = -\frac{1}{3}F^2\left\{(n+1)\mathcal{K}_{.k} + 3\mathcal{K}\mathcal{I}_k\right\}.$$

which is equivalent to

$$\mathcal{L}_{ijk|m}y^m + \mathcal{K}F^2C_{ijk} = -\frac{1}{3}F^2\left\{\mathcal{K}_{.i}h_{jk} + \mathcal{K}_{.j}h_{ik} + \mathcal{K}_{.k}h_{ij}\right\},$$

$$\mathcal{J}_{k|m}y^m + \mathcal{K}F^2\mathcal{I}_k = -\frac{1}{3}(n+1)F^2\mathcal{K}_{.k}.$$

form which we get the following

$$\mathcal{L}_{ijk|m}y^m + \mathcal{K}F^2C_{ijk} = \boxed{\frac{1}{n+1}\left\{\mathcal{J}_{i|0}h_{jk} + \mathcal{J}_{j|0}h_{ik} + \mathcal{J}_{k|0}h_{ij}\right\}} + \frac{\mathcal{K}F^2}{n+1}\left\{\mathcal{I}_ih_{jk} + \mathcal{I}_jh_{ik} + \mathcal{I}_kh_{ij}\right\}.$$

Where the boxed is *mean Landsberg curvature*.

Just use the equations marked (7.4), (7.5), we have

[21] Thirteenth Time of Lecture, Thursday, June 3, 2010

- Landsberg metric + $\mathcal{K} = \text{const}(\neq 0) \Rightarrow F$ is Riemannian metric. As $\mathcal{L}_{ijk|m}y^m + \mathcal{K}F^2C_{ijk} = 0$.
- Landsberg metric + $\mathcal{K}(x,y)(\neq 0)$(Scalar curvature) + dim $\geq 3 \Rightarrow$ Riemannian.
  In fact, we have

  Landsberg mettic + $\mathcal{K}(x,y)(\neq 0)$ + dim $= n \geq 3 \Rightarrow F$is Randers metric

$$(C - recducible, C_{ijk} = \frac{1}{n+1}\{\mathcal{I}_i\}h_{jk} + \mathcal{I}_jh_{ik} + \mathcal{I}_kh_{ij}).$$

**Lemma 28.1** (Lemma 7.1.1, Schur Lemma)**.** *$(M,F)$ is a Finsler manifold, with* $\dim M \geq 3$, *and $\mathcal{K}(P,y) = \mathcal{K}(x)$, $x \in M$(means isotropic on $M$). Then, $\mathcal{K}$ =* const.

*Remark* 28.1 (of the proof). • $\left[F^2\right]_{y^k} = 2FF_{y^k} = 2y_k = 2g_{jk}y^j$,
- $2y_k\mathcal{K}_{|j} = \mathcal{K}_{|k}g_{jp}y^p + \mathcal{K}_{|m}y^mg_{jk}$ by (7.11).

28.2.

**Proposition 28.1.** *$(M,F)$ is a Finsler manifold. With the following properties:*
- $\mathcal{K} = \mathcal{K}(x,y)$,
- *$\mathcal{S} = (n+1)cF$, $c = c(x)$ is a function on $M$. i.e., $F$ is of scalar curvature.*

*then,*

$$\mathcal{K} = \frac{3c_{x^m}y^m}{F} + \sigma, \quad \sigma = \sigma(x), x \in M.$$

*Proof.* By (5.42): $\mathcal{S}_{\cdot k|m}y^m - \mathcal{S}_{|k} = -\frac{1}{3}\left(2\mathcal{R}^m_{k\cdot m} + \mathcal{R}^m_{m\cdot k}\right) \overset{(7.3)}{=} -\frac{n+1}{3}\mathcal{K}_{\cdot k}F^2$.

On the other hand, by $\mathcal{S} = (n+1)cF \Rightarrow \mathcal{S}_{\cdot k|m}y^m - \mathcal{S}_{|k} = (n+1)\{c_{|m}y^mF_{\cdot k} - c_{|k}F\}$, $F_{|k} = 0$. $\Rightarrow$ (7.13) : $c_{|m}y^mF_{\cdot k} - c_{|k}F = -\frac{1}{3}\mathcal{K}_{\cdot k}F^2 \cdots \Rightarrow \sigma := \mathcal{K} - \frac{3c_{x^m}y^m}{F}$, see London Math, J. 2003, S.Zhen, Mo., Cheng. □

**Corollary 28.1** (Corollary 7.1.3)**.** *Let $F$ be an $n$-dimensional Finsler metric of scalar flag curvature with flag curvature $\mathcal{K} = \mathcal{K}(x,y)$. If $F$ has constant $\mathcal{S}$-curvature, i.e., $\mathcal{S} = (n+1)cF$, where $c$ =* const, *then $\mathcal{K} = \mathcal{K}(x)$ is a scalar function on $M$.*

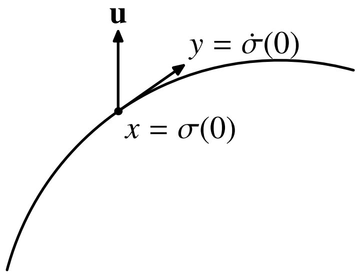
## 29. Global Rigidity Theorem

29.1. **Preliminary.**
- Define mean Matsumoto tensor at a point;
- $\mathcal{M}(xmr) := \sup_{\min(d(z,x),d(x,z))\leq r} \|\mathcal{M}\|_z = o(e^{kr})$, $r \to +\infty$. i.e.,

$$\lim_{r\to+\infty} \frac{\mathcal{M}(x,r)}{e^{kr}} = 0$$

  — *sub-exponentially grow* at rate of $k$.

**Theorem 29.1** (theorem 7.2.1)**.** *$(M,F)$ is a Finsler manifold, with* dim $\geq 3$, *and*




- *complete;*
- $\mathcal{K} = \mathcal{K}(x, y)$*;*
- $\mathcal{K} \leq -1$*;*
- *Matsumoto tension grows sub-exponentially at rate of* $k = 1$*. i.e.,* $\lim_{r\to+\infty} \frac{\mathcal{M}(x,r)}{e^r} = 0$*,* $\forall x \in M$*.*

*then,* $F = \alpha + \beta$*, is a Randers metric.*

*Proof.* Assume that $\mathcal{M}_y(u, u, u) \neq 0$, $y, u \neq 0$, with $F(x, y) = 1$.
**step 1.** Taking geodesic $\sigma = \sigma(t)$, $\infty < t < +\infty$, $\sigma(0) = x$, $\dot\sigma(0) = y$, by complete.
**step 2.** Let $u = u(t)$ is the linearly parallel of $\mathbf{u}$ along $\sigma$. i.e., $\mathrm{D}_{\dot\sigma}u(t) = 0$. Put $\mathcal{M}(t) := M_{\dot\sigma(t)}(u(t), u(t), u(t)) \Rightarrow \mathcal{M}''(t) + \mathcal{K}(t)\mathcal{M}(t) = 0$, which is the equation (7.15). Note $\mathcal{K}(t) := \mathcal{M}(\sigma(t), \dot\sigma(t)) \leq -1$, and consider $\mathcal{M}_0''(t) - \mathcal{M}_(t) = 0$, we have the solution $\mathcal{M}_0(t) = M(0)\cosh(t) + M'(0)\sinh(t)$.

Let $f(t) = \frac{\mathcal{M}'(t)}{\mathcal{M}(t)}$, then $f'(t) + f^2(t) = \frac{\mathcal{M}''(t)}{\mathcal{M}(t)} = -\mathcal{K}(t) \geq 1$. there, we note that $\mathcal{M}(0) \neq 0$, thus, we can pick an $\delta > 0$ small enough such that for any $|t| < \delta$, we have $\mathcal{M}(t) \neq 0$, and the above formula make sense.

Let $f_0(t) = \frac{\mathcal{M}_0'(t)}{\mathcal{M}_0(t)}$, note $\mathcal{M}_0(t) = M(0) \neq 0$, so we can make it have sense at an interval $\alpha < t\beta$. then, $f_0'(t) + f_0^2(t) = 1$.

Put $h(t) = (f(t) - f_0(t))\exp\left\{\int (f(t) + f_0(t))\,\mathrm{d}t\right\}$, then $h(0) = 0$, $h'(t) \geq 0$, so $h(t) \geq 0$ for $t \geq 0$, and $h(t) \leq 0$ for $t \leq 0$.

Put $\phi(t) = \left|\frac{\mathcal{M}(t)}{\mathcal{M}_0(t)}\right|$, $[\ln\phi(t)]' = \frac{\mathcal{M}_0(t)}{\mathcal{M}(t)}\phi'(t)$, by $\int \frac{1}{x}\mathrm{d}t = \ln|x| \Rightarrow [\ln\phi(t)]' = \frac{\mathcal{M}'(t)}{\mathcal{M}(t)} - \frac{\mathcal{M}_0'(t)}{\mathcal{M}_0(t)} = f(t) - f_0(t)\begin{cases} \leq 0, & t < 0 \\ \geq 0, & t > 0 \end{cases} \Rightarrow \phi'(t) \leq 0$, $t \leq 0$, and $\phi'(t) \geq 0$, $t > 0$. thus, $\phi(t)\phi(0) = 1$, $|\mathcal{M}(t)| \geq |\mathcal{M}_0(t)|$, $M(x, r) \geq \max\{\mathcal{M}(t)||t| \leq r\}$.

Note $M_0(t) = M(0)\cosh(t) + M'(0)\sinh()$, if $M'(0) = 0$ or $M'(0)M(0) > 0$, then $|M(x, r)| \geq |\mathcal{M}(r)| \geq |\mathcal{M}_0(r)| \geq |M(0)|\cosh(r) + |M'(0)|\sinh(r)$; If $M'(0)M(0) < 0$, then $M(x, r) \geq |\mathcal{M}(-r)| \geq |M(0)|\cosh(r) + |M'(0)|\sinh(r)$; Form both cases we can deduce

$$\liminf_{r\to\infty} \frac{M(x, r)}{e^r} \geq \frac{1}{2}\{|M(0)| + |M'(0)|\} > 0.$$

which is a contradiction with the grows rate is $k = 1$. □

**Theorem 29.2** (theorem 7.2.2)**.** *$(M, F)$ is a Finsler manifold, with*

- $\dim M \geq 3$*;*

- *closed;*
- $\mathcal{K} = \mathcal{K}(x, y)$*;*
- $\mathcal{K} < 0$.

*then* $F = \alpha + \beta$ *is a Randers metric.*

*Proof.* the proof in include in the proof of theorem 7.2.4($b$). □

*Remark* 29.1. Recall that the theorem of Akbar Zedah:

*Theorem* 29.3. *$(M, F)$ is a Finsler manifold, and*

- *closed;*
- $\mathcal{K} = \text{const}$*;*
- $\mathcal{K} < 0$.

*then $F$ is Riemannian.*

**Theorem 29.4** (theorem 7.2.4)**.** *Let $(M, F)$ be a complete Finsler manifold with isotropic flag curvature $\mathcal{K} = \mathcal{K}(x)$.*

*(1) If $\mathcal{K} \leq -1$ and $\mathcal{I}$ grows sub-exponentially at rate of $k = 1$, then $F$ if Riemannian.*
*(2) If $\mathcal{K} \leq 0$ and $C$ (resp.$\mathcal{I}$) grows sub-linearly, then $F$ is Landsbergian (resp. weakly Landsbergian). Further $F$ is Riemannian on any open subset where $\mathcal{K} < 0$.*

*Proof.* the proof is almost the same as theorem 7.2.1. □

## 30. RANDERS METRIC OF SCALAR FLAG CURVATURE

Progress:

- Z.Shen,2001: Projectively flat and $\mathcal{K} = \text{const}$(Not need be closed);
- Mo-Shen-Cheng. Projectively flat and $\mathcal{S} = (n + 1)c(x)F$. London Math, J. 2003. Note that $\mathcal{K} = \mathcal{K}(x, y) + \mathcal{S} = (n + 1)c(x)F \Rightarrow \mathcal{K} = \frac{3c_{x^m}y^m}{F} + \sigma(x)$.
- Bao-Robles-Shen 2003. Differential Geometry. $\mathcal{K} = \text{const} \Rightarrow \mathcal{K} = \mathcal{R}(x, y) + \mathcal{S} = (n + 1)c(x)F$ and Navigation problem.
- cheng-shen 2009.12 Aust. Math. J.

30.1. **navigation problem.** $F = \alpha + \beta \leftrightarrow (h, v)$, $F = \frac{\sqrt{\lambda h^2 + v_0^2}}{\lambda} - \frac{v_0}{\lambda}$. $h^2 = h^2(x, y)$, $v_0 = v_i y^i = h_{ij} v^j y^i$.

- $G^i = G^i h + \cdots$;
- $\mathcal{S} = \frac{n+1}{2F}\left\{2FR_0 - R_{00} - F^2 R\right\}$;

Let (P112. Lemma 8.1.4) $^i = G^i h + Q^i \Rightarrow$ (7.29), $R^i_k = \bar{R}^i_k + 2Q^i_{|k} - \left[Q^i_{|m}\right]_{y^k} y^m + 2Q^m \left[Q^i\right]_{y^m y^k} - \left[Q^i\right]_{y^m} [Q^m]_{y^k}$, or equally, $R^i_k = \bar{R}^i_k 2Q^i_{|k} - Q^i_{|m\cdot k} y^m + 2Q^m Q^i \cdot m \cdot k - Q^i_{\cdot m} Q^m_{\cdot k}$. Assume $\mathcal{S} = (n+1)cF$, of isotropy $\mathcal{S}$ curvature. $\Leftrightarrow \mathcal{R}_{ij} = -2ch_{ij}$. i.e., $v_{i|j} + v_{j|i} = -4ch_{ij} \Rightarrow \mathcal{R}_j = v^i \mathcal{R}_{ij} = -2cv_j$.

$Q^i := -F\mathcal{S}^i_0 - \frac{1}{2}F^2\mathcal{S}^i + cFy^i. \Rightarrow 2Q^i_{|k} = \cdot, Q^i_{\cdot k|0} = \cdot, 2Q^m Q^i_{\cdot m\cdot k} = \cdots, Q^i_{\cdot m}Q^i_{\cdot k} = \cdots$, all of this is a complex computation, and about 160 terms.

then you should verify the (7.32) $\Rightarrow$ (7.34).

**Lemma 30.1** (Lemma 7.3.1)**.** *Use 7.33 to get an proof.*

**Theorem 30.1** (7.3.2)**.** *$F = \alpha + \beta \leftrightarrow (N.P) \quad (h, v)$. Suppose that $\mathcal{S} = (n+1)F$, then $\mathcal{K} = \mathcal{K}(x, y) \Leftrightarrow \bar{\mathcal{K}} = \bar{\mathcal{K}}(x)$ sectional curvature of $h$.*

*Proof.* "$\Rightarrow$" $\mathcal{K} = \frac{2c_{x^m}y^m}{F} + \sigma \Leftrightarrow \mathcal{R}^i_k = \left(\frac{3c_{x^m}y^m}{F} + \sigma\right)$.

Note that $F^2\delta^i_k - FF_{y^k y^i} = F^2\left\{\delta^i_k - l_k l^i\right\}$.

Let $\mu := \sigma + c^2 + 2c_{x^m}v^m$, then by (7.34)

$\tilde{\mathcal{R}}^i k - \mu\left(\tilde{h}^2\delta^i_k - \xi_k\xi^i\right) - \frac{1}{\tilde{h}+\tilde{v}_0}\xi_k\left\{\tilde{\mathcal{R}}^i_p - \mu\left(\tilde{h}^2\delta^i_p - \xi_p\xi^i\right)\right\}v^p = 0 \Rightarrow \left(\tilde{\mathcal{R}}^i_k - \mu(\tilde{h}^2\delta^i_k - \xi_k\xi^i)\right)v^k = \frac{\boxed{\xi_k v^k = \tilde{v}_0}}{\tilde{h}+\tilde{v}_0}\cdot\left(\tilde{\mathcal{R}}^i_p - \mu(\tilde{h}^2\delta^i_p - \xi_p\xi^i)\right)v^p \Rightarrow \left(1 - \frac{\tilde{v}_0}{\tilde{h}+\tilde{v}_0}\right)\left(\tilde{\mathcal{R}}^i_p - \mu\left(\tilde{h}^2\delta^i_p - \xi_p\xi^i\right)\right) = 0 \Rightarrow \tilde{\mathcal{R}}^i_p - \mu\left(\tilde{h}^2\delta^i_p - \xi_p\xi^i\right) \Rightarrow \tilde{R}^i_k = \mu\left(\tilde{h}^2 - \delta^i_k - \xi_k\xi^i\right)$.

Note $\tilde{\mathcal{R}}^i_k = \bar{\mathcal{R}}^i_{jkl}\xi^j\xi^k$, $\bar{\mathcal{R}}^i_k = \bar{\mathcal{R}}^i_{jkl}y^j y^k$.

and finally we get $\mathcal{K} = \mu(x)$. □

30.2. **The Weak Einstein Metric**[22]**.** First recall that $F = \alpha + \beta \xleftrightarrow{\text{N.V.P}} (h, v)$; Assume that $\mathcal{S} = (n+1)cF \Rightarrow$ (7.34).

**Definition 30.1.** we call a metric is *Einstein metric* if **Ric** $= (n-1)c(x)F^2$. And call it is a *weak Einstein* if **Ric** $= (n-1)\left(\frac{3c_{x^m}y^m}{F} + \sigma(x)\right)F^2$.

If a Finsler metric is of scalar flag curvature and isotropic S-curvaturethen we know that **Ric** $= (n-1)\left(\frac{3c_{x^m}y^m}{F} + \sigma(x)\right)F^2$, that is $F$ is a weak Einstein metric.

Now, Let **Ric** $= \mathcal{R}^m_m$, $\overline{\textbf{Ric}} = \bar{\mathcal{R}}^m_m$, $\widetilde{\textbf{Ric}} := \tilde{\mathcal{R}}^m_m = \bar{\mathcal{R}}_p{}^m{}_{mq}\xi^p\xi^q$, there $\bar{\mathcal{R}}_p{}^m{}_{mq}$ is the Riemannian curvature of $h$.

**Lemma 30.2** (lemma 7.3.5, see [1])**.** *Let $F = \alpha + \beta \xleftrightarrow{\text{N.V.P}} (h, v)$. Assume that $\mathcal{S} = (n+1)cF$(isotropic* S-curvature). then

$$\textbf{Ric} - (n-1)\left(\frac{3c_{x^m}y^m}{F} + \mu - c^2 - 2c_{x^m}v^m\right) = \widetilde{\textbf{Ric}} - (n-1)\mu\tilde{h}^2.$$

where $\tilde{h}$ is just substitute $y$ in $h$ by $\xi$.

**Theorem 30.2** (theorem 7.3.6, see [1])**.** *Suppose $F = \alpha + \beta \xleftrightarrow{\text{N.V.P}} (h, v)$. Assume that $\mathcal{S} = (n+1)cF$. Then $F$ is weak Einstein metric, i.e.,* **Ric** $= (n-1)\left(\frac{3}{c_{x^m}y^m}F + \sigma\right)F^2 \Leftrightarrow \overline{\textbf{Ric}} = (n-1)\mu h^2$*. Where $\mu := \sigma + c^2 + 2c_{x^m}v^m$.*

*Proof.* Use Lemma 7.3.5 and the arbitrary of $\xi$ is equal to say the arbitrary of $y$. □

[22]The Fourteenth Time of Lecture, Thursday, 10 June, 2010

**Lemma 30.3** (lemma 7.3.7, see [1]). *Which is just said that if $F$ is of constant coefficient of Ric curvature, and is Einstein metric. Then $F$ must of constant* S-curvature. In particular, constant flag curvature implies that constant S-curvature.

30.3. **Randers metric of constant flag curvature.** The work of classification of constant flag curvature of Randers metric has done by D.Bao, Robles, Z.Shen, 2004. *Randers metric of flag curvature.*

It must to mention here that the related work has recently done by X.Y. Cheng, Z.Shen, 2005 has completed and published at 2009. *Randers metric of scalar flag curvature with isotropic* S-curvature. See theorem 7.3.3, it include the aforementioned work as a special case.

Now suppose that $F = \alpha + \beta \xleftrightarrow{\text{N.V.P}} (h, v)$. $F$ is of constant flag curvature $\mathcal{K}$ is equal to

- $R_{00} = -2ch^2$ (isotropic S-curvature), $c$ is a constant;
- $\bar{\mathcal{K}} = \mu =$ const, in this case, $\bar{\mathcal{K}} = \mathcal{K} + C^2$. (D.Bao, Robles).

and also equal to

- $\mathcal{K} =$ const;
- $\mathcal{S} = (n+1)cF$, $c =$ const. and also equal to (Cheng-shen)
  - $h = h\mu$;
  - $v =$ (7.45) given by (7.40). and
  - $c = \begin{cases} \delta, & \text{if } \mu = 0; \\ 0, & \text{if } \mu \neq 0. \end{cases}$

  In fact, $c(\mathcal{K} + c^2) = 0$, where $\bar{\mathcal{K}} = \mu$.

So the classification of cheng-shen include the classification of D.Bao, Robles.

## Part 8. Chapter 8, Projectively Flat Finsler metrics

$F$ is projectively flat means $F_{x^k y^l} y^k - F_{x^l} = 0$(Hamilton equation), and equally,

$$G^i = Py^i, \quad P = \frac{F_{x^k y^k}}{2F}.$$

### 31. Projectively Flat Randers Metrics

A Randers metric $F = \alpha + \beta$, is projectively flat if and only if

- $\alpha$ is of constant curvature;
- $\beta$ is closed.

31.1. **Projectively Flat Randers metric of constant flag curvature.** .

**Proposition 31.1** (proposition 8.1.1, see [1]). *$F = \alpha + \beta$ be a locally projectively flat Randers metric on $M$. Suppose that it has constant Ricci curvature* **Ric** $= (n-1)\lambda F^2$. *Then $\lambda \leq 0$. If*

*$\lambda = 0$, then $F$ is locally Minkowskian. If $\lambda = -1/4$, then $F$ is given by*

$$\boxed{F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} \pm \langle x, y\rangle}{1 - |x|^2}} \pm \frac{\langle \mathbf{a}, y\rangle}{1 + \langle \mathbf{a}, x\rangle},$$

*where $\mathbf{a} \in \mathbf{R^n}$ is a constant vector. In this case,*

$$\mathcal{K} = -\frac{1}{4}, \quad \mathcal{S} = \pm\frac{1}{2}(n+1)F.$$

*Note that the other values of $\lambda < 0$ is just this case with a conformal map.*

*and the boxed part is a Funk metric on unit sphere.*

*Proof.* By the assumption, $F$ is projectively flat. Let $\mathcal{K}_\alpha = \mu$ (= const), $\beta$ is closed. From (6.10) $\mathbf{Ric} = \overline{\mathbf{Ric}} + (n-1)\left[3\left(\frac{\Phi}{2F}\right)^2 - \frac{\Psi}{2F}\right]$. As $\mathbf{Ric}$, $\overline{\mathbf{Ric}}$ both are constant, $\mu\alpha^2 + 3\left(\frac{\Phi}{2F}\right)^2 - \frac{\Psi}{2F} = \lambda F^2$.

Use the fact that $\alpha$ can't be a polynomial(it is a irrational quadratical radical) and $\beta$ is homogenous polynomial to get (8.3) and (8.4). $\cdots \Rightarrow 4(\mu - 4\lambda)\Phi\alpha^2\beta^2 = 0$. $\Phi = b_{i;j}y^iy^j = 0 \Rightarrow b_{i;j} = 0 \Rightarrow \beta_{|j} = 0$. so $\Psi = b_{i;j;k}y^iy^jy^k = 0$, form (8.4) to get $(2\mu - 4\lambda)\alpha^2\beta = 4\lambda\beta^3$ on $\mathcal{U}$, as $\beta \neq 0$. $(2\mu - 4\lambda)\alpha^2 = 4\lambda\beta^2 \Rightarrow$ a contradiction (as if $\alpha^2 = k\beta^2$, $\beta^2 = b_ib_jy^iy^j = \mathbf{y}^T\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}(b_1\, b_2 \cdots b_n)\mathbf{y}$. The rank of $\alpha$ is $n$, while the rank of $\beta$ is 1.) $\Phi = b_{i;j}y^iy^j$, $e_{ij} = r_{ij} + s_ib_j + s_jb_i = r_{ij}$ ($\beta$ is closed, so $b_i$, $b_j$ is 0.), $e_{ij} = (b_{i;j} + b_{j;i})/2 \Rightarrow \cdots$. □

31.2. **projectively flat Randers metric with isotropic** S-curvature.(Cheng-Mo-Sheng).

**Theorem 31.1** (theorem 8.1.2 see [1]). *Projectively flat $\Rightarrow \alpha$ is of constant curvature and $\beta$ is closed. $F$ has almost isotropic s-curvature $\mathcal{S} = (n+1)\{cF + \eta\}$, $\eta$ is a closed 1-form. Then the flag curvature is $\mathcal{K} = \frac{3c_{x^m}(x)y^k}{F(x,y)} + 3c(x)^2 + \mu$, and also, $\mathcal{K} = \frac{3}{4}\left\{\mu + 4c(x)^2\right\}\frac{F(x,-y)}{F(x,y)} + \frac{\mu}{4}$. where $c_{x^k}$ denote the partial derivative of $c$ with respect to $x^k$. Moreover,*

- *if $\mu + 4c(x)^2 \equiv 0$, then $c(x) = c$ is a constant and the flag curvature $\mathcal{K} = -c^2$. In this case, $F = \alpha + \beta$ is either locally Minkowskian (if $c = 0$),or, up to a scaling ($c = \pm 1/2$), locally isometric to the metric in (8.2);*

  *In fact, this case is just the theorem 8.1.1.*
- *if $\mu + 4c(x)^2 \neq 0$, then $F = \alpha + \beta$ is locally given by*

$$\alpha \cong \alpha_\mu(x, y), \quad \beta = -\frac{2c_{x^k}y^k}{\mu + 4c(x)^2}.$$

*where $\alpha_\mu$ is given by (8.1) and $c(x) := c_\mu(x)$ is given by*

$$c_\mu(x) = \begin{cases} (\lambda + \langle \mathbf{a}, \mathbf{x}\rangle)\sqrt{\frac{\mu}{\pm(1+\mu|x|^2) - (\lambda + \langle \mathbf{a}, \mathbf{x}\rangle)^2}}, & \text{if } \mu \neq 0; \\ \frac{\pm 1}{2\sqrt{\lambda + 2\langle \mathbf{a}, \mathbf{x}\rangle + |x|^2}}, & \text{if } \mu = 0. \end{cases}$$

*where* $\mathbf{a} \in \mathbf{R}^n$ *is a constant vector and* $\lambda$ *is a constant number.*

## 32. Projectively flat metrics with constant flag curvature

32.1.

**Lemma 32.1** (lemma 8.2.1, see [1]). *$F = F(x, y)$ on $M$ is projectively flat with constant curvature $\mathcal{K} = \lambda$ if and only if there exist $P = P(x, y)$ such that $F_{x^k} = (PF)_{y^k}$; $P_{x^k} = PP_{y^k} - \lambda F F_{y^k}$. Where $P = \frac{1}{2F} F_{x^m} y^m$ is the projectively flat of $F$, and $P^2 - P_{x^k} y^k = \lambda F^2$.*

*Proof.* "$\Rightarrow$" $F$ is projectively flat, $F_{x^k y^l} y^k - F_{x^l} = 0$, $(PF)_{y^k} = \frac{1}{2}(F_{x^m} y^m)_{y^k} = \cdots = F_{x^k}$ by Hamilton equation.

Further more, $\mathcal{R}^i_k = \Xi \delta^i_k + \tau_k y^i$. $\Xi := P^2 - P_{x^k} y^k$, $\tau_k := 3\left(P_{x^k} - PP_{y^k} + \Xi_{\cdot k}\right)$.

$F$ is projectively flat, $\mathcal{K} = \frac{\Xi}{F^2} = \frac{P^2 - P_{x^k} y^k}{F^2}$. (Note this is by Berwald (1929), $P_{x^k} - PP_{y^k} = -\frac{(\Xi F)_{y^k}}{3F}$).

By $\Xi := \mathcal{K} F^2 = \lambda F^2 \Rightarrow$ (8.25). □

[23]Recall that we already give a classification of projectively flat Randers metric. Generally speaking, there are tow classes, one is of constant flag curvature, and the other is that of isotropic S-curvature. Now we'll turn to the more general case: to classify the general Finsler metric, and you will find there have almost the same two classes.

### 32.2. Projectively flat Finsler metric with negative constant curvature.

**Theorem 32.1** (theorem 8.2.3, see [1]). *Let $\psi = \psi(y)$ is a Minkowski norm. $\varphi = \varphi(y)$ is a 1(p)-homogenous function. Put $\Psi_\pm(x, y) = \phi_\pm(y + \Psi_\pm(x, y)x)$-just the same property as Funk metric. where, $\phi_\pm(y) := \varphi(y) \pm \psi(y)$. Define*

$$F(x, y) = \frac{1}{2}\left\{\Psi_+(x, y) - \Psi_-(x, y)\right\},$$

*then, $F$ is projectively flat and*

- *$\mathcal{K} = -1$;*
- *Projectively factor is $P(x, y) = \frac{1}{2}\{\Psi_+(x, y) + \Psi_-(x, y)\}$;*
- *$P(0, y) = \varphi(y)$ and $F(0, y) = \psi(y)$.*

*Remark* 32.1. Recall that the *Funk metric* is defined on a convex domain $\Omega$ with $x + \frac{y}{\Theta(x,y)} \in \partial\Omega$. $\phi\left(y + \frac{y}{\Theta(x,y)}\right) = 1 \Leftrightarrow \Theta(x, y) = \phi(y + x\Theta(x, y)$. And then you'll find the above theorem is quite the same as Funk metric. and also the latter one (theorem 8.2.7, see [1]).

the proof should use the following lemma

[23]The Fifteenth Time of Lecture, Thursday, 13 June, 2010

**Lemma 32.2** (Lemma 8.2.4, see [1]). *Let $\phi = \phi(y)$ is an arbitrary positively homogenous function of degree one on $\mathbf{R^n}$. And $\phi$ is $C^\infty$ on $\mathbf{R^n} \setminus \{\mathbf{0}\}$. Then for $x$ close to the origin, there is a unique real-valued function $f := f(x, y)$ such that*

$$f(x, y) = \phi\left(y + f(x, y)x\right),$$

*and*

$$f_{x^k} = f f_{y^k}.$$

*Proof.* By Assumption $\phi(y)$ is $C^\infty$ so $\phi_{y^i}(y)$ is also $C^\infty$. So we can suppose $|\phi_{y^i}| \leq M$ for some positive $M$ on $\mathbf{R^n}$. taking $\delta \leq \frac{1}{2M}$. then when $|x| < \delta$, we have $|\varphi_{y^i} x^i| \leq \|\varphi_{y^i}\| \cdot \|x^i\| \leq M \cdot \delta = \frac{1}{2}$. thus $\varphi_{y^i} x^i \leq \frac{1}{2}$.

Let $h(t) := t - \phi(y + tx)$, then $h'(t) = 1 - \phi_{y^k}(y + tx)x^k \geq \frac{1}{2}$. $\cdots$ □

*proof of theorem 8.2.3.* The key point is equation (8.33): $(\Psi_\pm)_{x^k} = \Psi_\pm (\Psi_\pm)_{y^k}$, use this to verify that $F_{x^k y^l} y^k - F_{x^l} = 0 \Rightarrow F$ is Projectively flat, and $P := \frac{F_{x^k} y^k}{2F}$, $\mathcal{K} = \frac{P^2 - P_{x^k} y^k}{F^2} = -1$. □

*Example* 32.1. Examples, these examples is just special case of the theorem, and you should verify it carefully.

### 32.3. Projective flat Finsler metrics with zero flag curvature.

**Theorem 32.2** (theorem 8.2.7, see [1]). *Let $\psi = \psi(y)$ is a Minkowski norm, $\varphi = \varphi(y)$ is (1)p-homogenous. Define $P \overset{\text{def}}{=} P(x, y) := \varphi\left(y + P(x, y)x\right)$. Let $F := \psi\left(y + P(x, y)x\right)\left\{1 + P_{y^m}(x, y)x^m\right\}$. then, $F$ is projectively flat and*

- *$\mathcal{K} = 0$;*
- *$P = P(x, y)$;*
- *$F(0, y) = \psi(y)$, $P(0, y) = \varphi(y)$.*

The proof and examples should study by yourself.

### 32.4. Projectively flat Finsler metrics with positive flag curvature.

This case involve some matters about complex analysis (although maybe not hard as you think), and we will discuss it next term. The main tool is power explanation, not the same as the aforementioned cases.

## 33. Projectively flat metrics with almost isotropic S-curvature

**Proposition 33.1** (Proposition 8.3.1, see [1]). *Let $F = F(x, y)$ is projectively flat, $\mathcal{S} = (n + 1)\{cF + \eta\}$, there $\eta$ is a closed 1-form (Note that the almost isotropic S-curvature is just use a different volume form). Then,*

- *If $\mathcal{K} \neq -c^2 + \frac{c_{x^m} y^m}{F} \Rightarrow F = \alpha + \beta$;*

- *If* $\mathcal{K} \equiv -c^2(x) + \frac{c_{x^m} y^m}{F(x,y)} \Rightarrow c(x) = c$ *is constant, and* $F = \begin{cases} \textit{Locally Minkowski}, & \text{if} c = 0; \\ \Theta(x,y) + \frac{\langle \mathbf{a}, y\rangle}{1+\langle \mathbf{a}, y\rangle}, & \text{if} c = \frac{1}{2}; \\ \Theta(x,-y) - \frac{\langle \mathbf{a}, y\rangle}{1+\langle \mathbf{a}, y\rangle}, & \text{if} c = -\frac{1}{2}. \end{cases}$
*where* $\mathbf{a} \in \mathbf{R^n}$ *is a constant vector and* $\Theta = \Theta(x,y)$ *is the Funk metric (define by 1.3.8, see* [1]*).*

*Remark* 33.1. Note that the case with $c \neq 0$ is Funk metric plus an one-form, or its symmetry form. There is an thesis of Okada on *Tenser*, 1981. give an apply of this proposition.

*Proof.* **step 1.** $F$ is projectively flat, $G^i = Py^i$, $P := \frac{F_{x^k} y^k}{2F}$. and $\mathcal{K} = \frac{P^2 - P_{x^k} y^k}{F^2}$. **step 2.** S−curvature $\mathcal{S} = (n+1)\{cF + \eta\}$, $\eta$ is a closed one-form, suppose $\eta = \mathrm{d}h$. then, $\mathcal{S} = \frac{\partial G^m}{\partial y^m} - \frac{y^m}{\sigma_F}\frac{\partial \sigma_F}{\partial x^m} = (n+1)P - \boxed{y^m \frac{\partial (\ln \sigma_F)}{\partial y^m} = \mathrm{d}(\ln \sigma_F)(y)} \Rightarrow P = cF + \mathrm{d}\varphi(y)$, where $\phi = h(x) + \frac{1}{1+n}\ln[\sigma_F(x)]$. Further, by $P \Rightarrow \mathcal{K} = \frac{P^2 - P_{x^k} y^k}{F^2} = (8.56)$. **step 3.** on the other hand, $\mathcal{K} = 3\frac{c_{x^m} y^m}{F} + \sigma$(proposition 7.12, see [1]). $\Rightarrow$ (8.58). Note the highest term of $F$ with coefficient $(\sigma + c^2)$.

Now if $\mathcal{K} \neq -c^2(x) + \frac{c_{x^m} y^m}{F} \cdots \Rightarrow F(x_0, y) = \frac{\{\phi_{x^i x^j}(x_0) - \phi_{x^i}(x_0)\phi_{x^j}(x_0)\} y^i y^j}{4 c_{x^m}(x_0) y^m} = \frac{\alpha_0^2(y)}{\beta_0(y)}$. there, $\alpha_0$ is a Riemannian metric. and $\beta_0$ is a 1-form. (and the $\{\cdot\}$ is the Laplacian operator, not change the sign.) $F(x,y)$ is called *Kropina metric*, which has singular point, this is a contradiction with $F(x_0, y)$ is a Minkowski metric on $T_{x_0}M$. Thus, $\sigma + c^2 \neq 0$. By (8.58),

$$F = \frac{\sqrt{(\sigma + c^2)(\varphi_{x^i x^j} - \varphi_{x^i}\varphi_{x^j}) y^i y^j + 4(c_{x^m} y6m)^2} - 2 c_{x^m} y^m}{\sigma + c^2}$$

is a Randers metric.

Now if $\mathcal{K} \equiv -c^2(x) + \frac{c_{x^i}(x) y^i}{F(x,y)} \Rightarrow \sigma(x) + c^2(x) + \frac{2 c_{x^m}(x) y^m}{F} \equiv 0 \Rightarrow c(x) = c\,(= \text{const}), \sigma(x) = -c^2$. and $\mathcal{K} = -c^2 \Rightarrow \varphi_{x^i x^h} - \varphi_{x^i}\varphi_{x^j} = 0 \Leftrightarrow \frac{\varphi_{x^i x^j}}{\varphi_{x^i}} = \varphi_{x^j} \Leftrightarrow \frac{\partial \ln \varphi_{x^i}}{\partial x^j} = \varphi_{x^j} \Leftrightarrow \ln \varphi_{x^i} = \varphi + a_i \Leftrightarrow \varphi = -\ln(1 + \langle \mathbf{a}, x\rangle) + C. \cdots$ □

The last words: Work hard on it, and read the Okaba's article, and wight Course Papers. See you next term!

# **Part** 9. **Reference and Index**

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SOUTHWEST UNIVERSITY · SCHOOL OF MATHEMATICS AND STATISTICS
*E-mail address*: van141.abel@gmail.com