Integral Geometry & Geometric Probability

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URL of Beamer Slides: *"Integral Geometry and Geometric Probability"*

`http://www.math.utah.edu/treiberg/IntGeomSlides.pdf`

Some excellent references to Integral Geometry.

- *Luis A. Santaló,* Integral Geometry and Geometric Probability, *Addison-Wesley, Reading, MA, 1976.*
- *Herbert Solomon,* Geometric Probability *(CBMS-NSF Regional Conference Series in Applied Mathematics 28), Society for Industrial and Applied Mathetaics, Philadelphia, 1978.*
- *Wilhelm Blaschke,* Vorlesungen über Integralgeometrie I, II, *Chelsea, New York, 1949, (2nd ed. orig. pub. B. G. Teubner, Leipzig, 1935.)*

3. Outline.

Integral Geometry, known in applied circles as Geometric Probability, is somewhat of a mathematical antique (and therefore it is a favorite of mine!) From it developed many modern topics: geometric measure theory, stereometry, tomography, characteristic classes. . .

1. Integral geometry examples:
   - Buffon's needle problem.
   - Firery's dice problem
2. Kinematic measure.
3. Poincaré's Formula for average number of intersections of curves.
4. Cauchy's Formula for the average projected length.
5. Crofton's Formula for the average chord length.
6. Santaló's & Blaschke's Formuls for the averages over the of the intesection of two domains.
7. Application to the Isoperimetric Inequality.

Theorem (Buffon's Needle Problem [1733])

Parallel lines on a wooden floor are a distance d apart from each other. A needle of length ℓ ($\ell < d$) is randomly dropped onto the floor. Then the probability that the needle will touch one of the lines is

$$P = \frac{2\ell}{\pi d}.$$

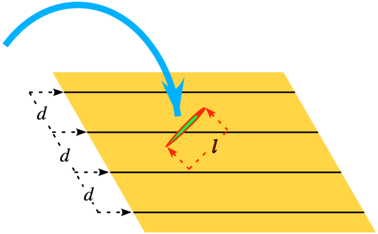

Figure: Buffon's Needle is randomly dropped

Theorem (Firey's Colliding Dice Problem [1974])

Suppose Ω_1 and Ω_2 are disjoint unit cubes in $\mathbf{R}^3$. In a random collision, the probablity that the cubes collide edge-to-edge slightly exceeds the probability that the cubes collide corner-to-face. Indeed,

$$0.54 \cong P(\textit{collide edge-to-edge}) > P(\textit{collide corner-to-face.}) \cong 0.46.$$

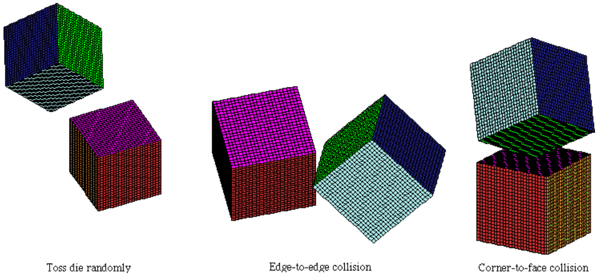

Figure: Almost all random cube collisions are edge-to-edge or corner-to-face.

6. Coordinates of a line.

An unoriented line in the plane is determined by two numbers, p the distance to the origin and θ, the direction to the closest point.
The variable range is $0 \leq p$ and $0 \leq \theta < 2\pi$.
Equivalently, we may take the range $-\infty < \tilde{p} < \infty$ and $0 \leq \eta < \pi$.

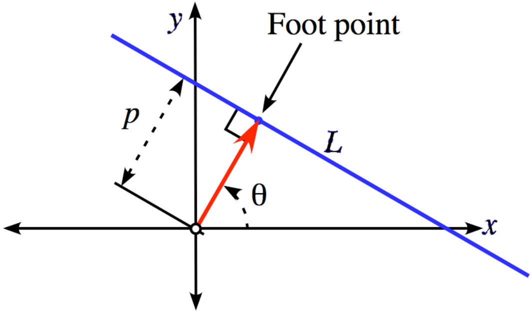

Figure: (p, θ) coordinates for the line L.

The equation of the line $L(p, \theta)$ in Cartesian coordinates is

$$\cos(\theta)x + \sin(\theta)y = p \qquad (1)$$

A rigid motion $\mathcal{M}$ of a set of points is given by a rotation by an angle α followed by a translation by the vector (x_0, y_0). Thus

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathcal{M} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus the inverse motion is therefore given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' - x_0 \\ y' - y_0 \end{pmatrix} \tag{2}$$

The mobile line $L(p, \theta)$ may be thought of as moving the fixed line $L(0, 0)$ by the translation $(x, y) \mapsto (x + p, y)$ followed by the rotation about the origin by angle θ.

The first task is to find a measure on a set of lines that is invariant under rigid motions. This measure will be called KINEMATIC MEASURE.

8. Kinematic measure.

The kinematic measure for lines in (p, θ) coordinates is given by

$$dK = dp \wedge d\theta.$$

To check that this measure is invariant under rigid motions, let us first determine how (p, θ) in the equation of the line (1) is changed by a rigid motion $\mathcal{M}$. We express (x, y) in terms of (x', y') using (2)

$$\begin{aligned} p &= \cos(\theta)x + \sin(\theta)y \\ &= \cos(\theta)\big[\cos(\alpha)(x' - x_0) + \sin(\alpha)(y' - y_0)\big] \\ &\quad + \sin(\theta)\big[-\sin(\alpha)(x' - x_0) + \cos(\alpha)(y' - y_0)\big] \\ &= [\cos\theta \, \cos\alpha - \sin\theta \, \sin\alpha]\,(x' - x_0) \\ &\quad + [\cos\theta \, \sin\alpha + \sin\theta \, \cos\alpha]\,(y' - y_0) \\ &= \cos(\theta + \alpha)(x' - x_0) + \sin(\theta + \alpha)(y' - y_0) \end{aligned}$$

or the equation of the new line L' becomes

$$p + \cos(\theta + \alpha)x_0 + \sin(\theta + \alpha)y_0 = \cos(\theta + \alpha)x' + \sin(\theta + \alpha)y'.$$

Thus we read off the (p', θ') coordinates of the line $L' = \mathcal{M}(L)$.

$$\begin{aligned} p' &= p + \cos(\theta+\alpha)x_0 + \sin(\theta+\alpha)y_0 \\ \theta' &= \theta + \alpha. \end{aligned}$$

Then the Jacobian formula for the change in measure is

$$dp' \wedge d\theta' = |J|\, dp \wedge d\theta$$

where

$$J = \frac{\partial(p', \theta')}{\partial(p, \theta)} = \begin{vmatrix} \dfrac{\partial p'}{\partial p} & \dfrac{\partial p'}{\partial \theta} \\[2ex] \dfrac{\partial \theta'}{\partial p} & \dfrac{\partial \theta'}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 1 & * \\ 0 & 1 \end{vmatrix} = 1.$$

Thus we have shown that the kinematic measure is invariant under rigid motions. □

We view (p', θ') as function (p, θ). The differentials are thus

$$
\begin{aligned}
dp' &= dp + \left\{ -sin(\theta + \alpha) x_0 + cos(\theta + \alpha) y_0 \right\} d\theta, \\
d\theta' &= d\theta.
\end{aligned}
$$

Recall that wedge is a skew product so that $dp \wedge d\theta = -d\theta \wedge dp$ and $d\theta \wedge d\theta = 0$. Hence

$$
\begin{aligned}
dp' \wedge d\theta' &= \left(dp + \left\{ -sin(\theta + \alpha) x_0 + cos(\theta + \alpha) y_0 \right\} d\theta \right) \wedge d\theta \\
&= dp \wedge d\theta.
\end{aligned}
$$

Let C be a piecewise $\mathcal{C}^1$ curve or network (a union of $\mathcal{C}^1$ curves.) Given a line L in the plane, let $n(L \cap C)$ be the number of intersection points. If C contains a linear segment and if L agrees with that segment, $n(C \cap L) = \infty$. For any such C, however, the set of lines for which $n = \infty$ has dK-measure zero.

Figure: Henri Poincaré 1854–1912

Theorem (Poincaré Formula for lines [1896])

Let C be a piecewise $\mathcal{C}^1$ curve in the plane. Then the measure of unoriented lines meeting C, counted with multiplicity, is given by

$$2\,\mathrm{L}(C) = \int_{\{L : L \cap C \neq \emptyset\}} n(C \cap L)\, dK(L).$$

For simplicity we assume C is a $\mathcal{C}^1$ curve $Z(s) = \big(x(s), y(s)\big)$, parameterized by arclength. Thus there are $x(s), y(s) \in \mathcal{C}^1[0, s_0]$ such that the tangent vector $\dot{Z} = (\dot{x}, \dot{y})$ satisfies $|\dot{Z}| = 1$. By adding the formulas for $\mathcal{C}^1$ curves gives the formula for integrating a piecewise $\mathcal{C}^1$ curve.

Let us consider a flag which is the set of pairs (L, Z) where L is a line in the plane and $Z \in L$ is a point. The set of lines and corresponding points that touch C gives the subset of the flag

$$\mathcal{S} = \{(L, Z); L \cap C \neq \emptyset, \quad Z \in L \cap C\}.$$

The line is determined by the coordinates (p, θ) and the point $Z \in L$ by an arclength coordinate q along L from the foot-point $(p\cos\theta, p\sin\theta)$.

$$\int_{\{L : L \cap C \neq \emptyset\}} n \, dK = \int_{\{L : L \cap C \neq \emptyset\}} \left(\sum_{Z \in L \cap C} 1 \right) dK \qquad (3)$$

13. Compute the integral of $\mathcal{S}$ in different coordinates.

On the other hand, the set $\mathcal{S}$ can be determined by the point $(x, y) = Z \in C$ first and then L can be any unoriented line through Z of angle $0 \le \eta < \pi$ (positive and negative orientations give the same line). Thus we may replace (p, θ) by the coordinates (s, η). Using

$$\tilde{p} = x(s)\,\cos\eta + y(s)\,\sin\eta.$$

$(\tilde{p}, \eta) \in (-\infty, \infty) \times [0, \pi)$ are same lines as $(p, \theta) \in [0, \infty) \times [0, 2\pi)$. So

$$d\tilde{p} = \left\{\dot{x}(s)\,\cos\eta + \dot{y}(s)\,\sin\eta\right\} ds + \left\{-x(s)\,\sin\eta + y(s)\,\cos\eta\right\} d\eta.$$

Changing to (s, η), using tangent direction $(\dot{x}, \dot{y}) = (\cos\phi(s), \sin\phi(s))$,

$$\begin{aligned} d\tilde{p}\,d\eta &= \left|\begin{vmatrix} \frac{\partial\tilde{p}}{\partial s} & \frac{\partial\tilde{p}}{\partial\eta} \\ \frac{\partial\eta}{\partial s} & \frac{\partial\eta}{\partial\eta} \end{vmatrix}\right| ds\,d\eta = \left|\begin{vmatrix} \cos\phi\,\cos\eta + \sin\phi\,\sin\eta & * \\ 0 & 1 \end{vmatrix}\right| ds\,d\eta \\ &= |\cos(\phi(s) - \eta)|\,ds\,d\eta. \end{aligned}$$

Using Fubini's theorem (slicing formula), we may reverse the order of integtation in (3) over the set $\mathcal{S}$,

$$\begin{aligned}\int_{\{L:L\cap C\neq\emptyset\}} \left(\sum_{Z\in L} 1\right) dK &= \int_{\{Z:Z\in C\}} \int_{\{L:Z\in L\}} d\tilde{p}\, d\eta \\ &= \int_0^{s_0} \int_0^{\pi} |\cos(\phi(s)-\eta)| d\eta\, ds \\ &= 2\int_C ds \\ &= 2\,\mathrm{L}(C).\end{aligned}$$

□

A nonempty set $\Omega \subset \mathbf{R}^2$ is convex if for every pair of points $P, Q \in \Omega$, the line segment $\overline{PQ} \subset \Omega$. The integral geometric formulas hold for convex sets. Since $n(L \cap \partial\Omega)$ is either zero or two for dK-almost all L, the measure of unoriented lines that meet the a convex set is given by

$$\mathrm{L}(\partial\Omega) = \int_{\{L : L\cap\Omega\neq\emptyset\}} dK.$$

The conditional probability of an event A given the event B is defined to be $P(A|B) = \frac{P(A\cap B)}{P(B)}$.

Theorem (Sylvester's Problem [1889])

Let $\omega \subset \Omega$ be two bounded convex sets in the plane. Then the probability that a random line meets ω given that it meets Ω is

$$P = \frac{\mathrm{L}(\partial\omega)}{\mathrm{L}(\partial\Omega)}.$$

Corollary

Let C be a piecewise $\mathcal{C}^1$ curve contained in a compact convex set Ω. Of all random lines that meet Ω, the expected number of intersections with with C is

$$\mathbb{E}(n) = \frac{2\,\mathrm{L}(C)}{\mathrm{L}(\partial\Omega)}. \tag{4}$$

Hence, there are *lines that cut C in at least $2\,\mathrm{L}(C)/\,\mathrm{L}(\partial\Omega)$ points.*

Proof. Since Ω is convex, $\mathbb{E}(n) = \dfrac{\int_{\{L:L\cap C\neq\emptyset\}} n\,dK}{\int_{\{L:L\cap\Omega\neq\emptyset\}} dK} = \dfrac{2\,\mathrm{L}(C)}{\mathrm{L}(\partial\Omega)}$.

The maximum of n exceeds the average. □

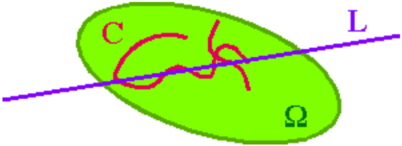

Figure: Average number of intersections $L \cap C$ of a line L meeting Ω.

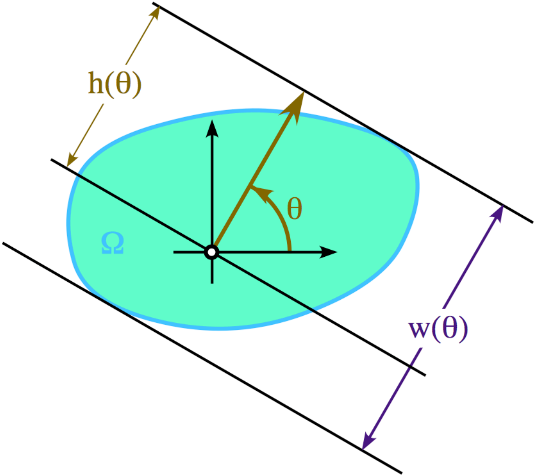

Figure: Width and support function of convex Ω in θ direction.

For $\theta \in [0, 2\pi)$, the support function, $h(\theta)$, is the largest p such that $L(p, \theta) \cap \Omega \neq \emptyset$. The width is $w(\theta) = h(\theta) + h(\theta + \pi)$.

Figure: Augustin Louis Cauchy 1789–1857

Theorem (Cauchy's Formula [1841])

Let Ω be a bounded convex domain. Then

$$\mathrm{L}(\partial\Omega) = \int_0^{2\pi} h(\theta)\, d\theta = \int_0^{\pi} w(\theta)\, d\theta. \quad (5)$$

$$\mathrm{L}(\partial\Omega) = \int_{\{L : L\cap\Omega\neq\emptyset\}} dK = \int_0^{2\pi}\int_0^{h(\theta)} dp\, d\theta$$
$$= \int_0^{2\pi} h(\theta)\, d\theta = \int_0^{\pi} h(\theta) + h(\theta+\pi)\, d\theta$$
$$= \int_0^{\pi} w(\theta)\, d\theta. \quad \square$$

Theorem

Suppose Ω is a compact, convex domain with a $\mathcal{C}^2$ boundary. Then

$$A(\Omega) = \frac{1}{2}\int_0^{2\pi} h\,ds = \frac{1}{2}\int_0^{2\pi} h(h+\ddot{h})\,d\theta. \tag{6}$$

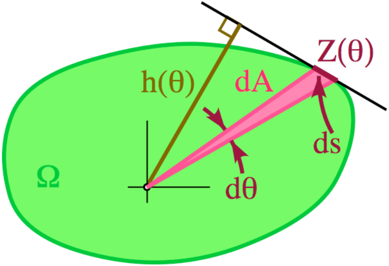

Figure: Area on polar coordinates.

Write $Z(\theta)$ for the point $L(h(\theta),\theta)\cap\partial\Omega$. The outer normal is $\mathbf{n}(\theta) = (\cos\theta, \sin\theta)$.

$$Z(\theta)\bullet\mathbf{n}(\theta) = h(\theta)$$

Since $\dot{\mathbf{n}} = (-\sin\theta, \cos\theta)$, and $\dot{Z}$ is tangent, $\dot{h} = \dot{\mathbf{n}}\bullet Z + \mathbf{n}\bullet\dot{Z} = \dot{\mathbf{n}}\bullet Z$. Thus $Z = h\mathbf{n} + \dot{h}\dot{\mathbf{n}}$. Hence, $\dot{Z} = \dot{h}\mathbf{n} + h\dot{\mathbf{n}} + \ddot{h}\dot{\mathbf{n}} - \dot{h}\mathbf{n} = (h+\ddot{h})\dot{\mathbf{n}}$.

Thus $\dfrac{ds}{d\theta} = h + \ddot{h}$ so $A(\Omega) = \displaystyle\int_\Omega dA$

$$= \frac{1}{2}\int_0^{2\pi} h\,ds = \frac{1}{2}\int_0^{2\pi} h(h+\ddot{h})\,d\theta. \qquad \square$$

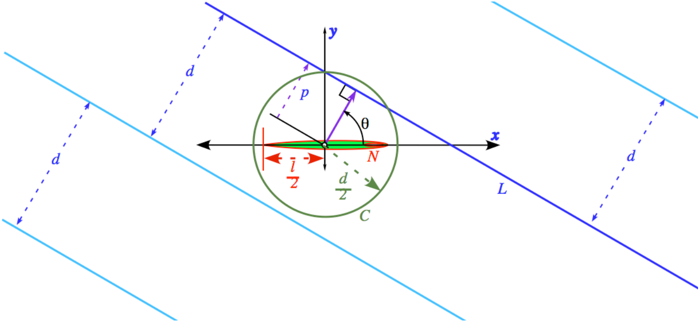

Figure: (p, θ) coordinates for the closest crack L.

Fix needle N, a line segment of length ℓ centered at origin. Move floor. $\ell < d$ implies only the cracks closest to the origin could touch the needle. So we consider crack lines L so that $\mathrm{dist}(L, 0) \leq \dfrac{d}{2}$ iff $C \cap L \neq \emptyset$, where C the circle about the origin with radius $\dfrac{d}{2}$.

Note that if $L \cap N \neq \emptyset$ then $n(L \cap N) = 1$. The probability of needle hitting a crack is

$$P = \frac{\int_{\{L : L \cap N \neq \emptyset\}} n(L \cap N)\, dK(L)}{\int_{\{L : L \cap C \neq \emptyset\}} dK(L)} = \frac{2\,\mathrm{L}(N)}{\mathrm{L}(C)} = \frac{2\ell}{2\pi \cdot \frac{d}{2}} = \frac{2\ell}{\pi d}. \qquad \square$$

An experimental determination of π.

$$\pi = \frac{2\ell}{Pd} \approx \frac{2\ell}{d} \cdot \frac{n}{x},$$

where x is the number of times needle touches crack in n trials.
Wolf, in Zurich (1850), tossed 5000 needles and found $\pi \approx 3.1596$.
A Scotsman, Smith (1855), repeated with $n = 3204$ and found $\pi \approx 3.1553$.

Figure: Morgan William Crofton 1826–1915.

Theorem (Crofton's Formula [1868])

Let $D \subset \mathbf{R}^2$ be a domain with compact closure, $L \subset \mathbf{R}^2$ a random line and $\sigma_1(L \cap D)$ be the length (one-dimensional measure). Then

$$\pi\, \mathrm{A}(D) = \int_{\{L : L \cap D \neq \emptyset\}} \sigma_1(L \cap D)\, dK(L).$$

Let the subset of the flag be
$\mathcal{S} = \{(L, Z) : L \cap D \neq \emptyset, \quad Z \in L \cap D\}$.
A point in $\mathcal{S}$ is given by coordinates (p, θ) describing the line and q, arclength in L from the foot point.

Denote the right side by $\mathcal{I}$. By extending $-\infty < \tilde{p} < \infty$, we double-count the lines.

$$\begin{aligned}
\mathcal{I} &= \int_{\{L : L\cap D\neq\emptyset\}} \sigma_1(L\cap D)\, dK(L) \\
&= \int_{\{L : L\cap D\neq\emptyset\}} \left(\int_{D\cap L} dq \right) dp\, d\theta \\
&= \int_0^{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \chi_{D\cap L}(q)\, dq\, dp\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{D\cap L}(q)\, dq\, d\tilde{p}\, d\theta
\end{aligned}$$

where $\chi_{D\cap L}$ is the characteristic function:

$$\chi_{D\cap L}(q) = \begin{cases} 1, & \text{if } q \in D\cap L; \\ 0, & \text{if } q \notin D\cap L. \end{cases}$$

24. Finish the proof of Crofton's Formula.

Observe that for the line $L(\tilde{p}, \theta)$ we have $\chi_{D \cap L}(q) = 1$ if and only if the point in the plane corresponding to $(\tilde{p}, q)$ lies in D, namely

$$\begin{aligned}(x, y) &= \tilde{p}(\cos\theta, \sin\theta) + q(-\sin\theta, \cos\theta) \\ &= (\tilde{p}\cos\theta - q\sin\theta, \tilde{p}\sin\theta + q\cos\theta) \in D\end{aligned}$$

thus

$$\chi_{L(\tilde{p},\theta)\cap D}(q) = \chi_D(x, y).$$

The change of variables to (x, y) is just rotation by angle θ. Thus

$$dx \wedge dy = \left[cos(\theta)d\tilde{p} - \sin(\theta)dq\right] \wedge \left[sin(\theta)d\tilde{p} + \cos(\theta)dq\right] = d\tilde{p} \wedge dq.$$

Now we think of $\mathcal{S}$ another way. First pick $Z \in D$ and then L is any line through Z.

$$\begin{aligned}
\mathcal{I} &= \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{D\cap L}(q)\,dq\,d\tilde{p}\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{D}(x,y)\,dx\,dy\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \mathrm{A}(D)\,d\theta \\
&= \pi\,\mathrm{A}(D).
\end{aligned}$$

□

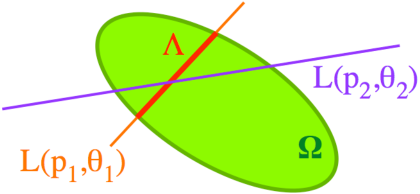

Figure: Two random lines that meet Ω

Corollary (Crofton [1885])

Let Ω be a bounded convex domain in the plane. Then the probability that two random lines intersect in Ω given that they both meet Ω is

$$P = \frac{2\pi\, \mathrm{A}(\Omega)}{\mathrm{L}(\partial\Omega)^2}.$$

By the isoperimetric inequality, $4\pi\, \mathrm{A}(\Omega) \le \mathrm{L}(\partial\Omega)^2$ with equality only for circle, the probability satisfies

$$P \le \frac{1}{2}.$$

Equality holds iff Ω is a round disk.

27. Compute the expected number of intersections of two lines.

Proof. Let $L_1(p_1, \theta_1)$ and $L_2(p_2, \theta_2)$ be two random lines whose invariant measure is $dK_1 \wedge dK_2 = dp_1 \wedge d\theta_1 \wedge dp_2 \wedge d\theta_2$.
View $\Lambda_1 = L(p_1, \theta_1) \cap \Omega$ as a subset. By (4), the average number of times that a random line $L(p_2, \theta_2)$ meets Λ_1 given that it meets Ω is

$$\mathbb{E}(n) = \frac{2\sigma_1\big(\Omega \cap \mathrm{L}(p_1, \theta_1)\big)}{\mathrm{L}(\partial\Omega)}.$$

Poincaré's and Crofton's Formulæ $\Longrightarrow$ probability that two lines meet is

$$\begin{aligned}
P = \mathbb{E}(n) &= \frac{\int_{\{L_1 : L_1 \cap \Omega \neq \emptyset\}} \int_{\{L_2 : L_2 \cap \Omega \neq \emptyset\}} \mathrm{n}(\Lambda_1 \cap L_2)\, dK_2\, dK_1}{\int_{\{L_1 : L_1 \cap \Omega \neq \emptyset\}} \int_{\{L_2 : L_2 \cap \Omega \neq \emptyset\}} dK_2\, dK_1} \\
&= \frac{\int_{\{L_1 : L_1 \cap \Omega \neq \emptyset\}} \mathbb{E}(n)\, dK_1}{\int_{\{L_1 : L_1 \cap \Omega \neq \emptyset\}} dK_1} = \frac{2\int_{\{L_1 : L_1 \cap \Omega \neq \emptyset\}} \sigma_1\big(\Omega \cap \mathrm{L}(p_1, \theta_1)\big)\, dK_1}{\mathrm{L}(\partial\Omega) \int_{\{L_1 : L_1 \cap \partial\Omega \neq \emptyset\}} dK_1} \\
&= \frac{2\pi\, \mathrm{A}(\Omega)}{\mathrm{L}(\partial\Omega)^2}. \quad \square
\end{aligned}$$

What is the average length of a chord of a compact convex set Ω?

28. Bertrand Paradox.

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There are many answers. Depends on what "random line" means.

What is the average length of a chord of a compact convex set Ω?
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1. Uniform distance from origin and uniform angle (proportional to dK)

$$\mathbb{E}(\sigma_1) = \frac{\int_{\{L : L \cap \partial\Omega \neq \emptyset\}} \sigma_1 \, dK}{\int_{\{L : L \cap \partial\Omega \neq \emptyset\}} dK} = \frac{\pi \, \mathrm{A}(\Omega)}{\mathrm{L}(\partial\Omega)}$$

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2. Uniform point on boundary and uniform angle
$$\mathbb{E}_2(\sigma_1) = \frac{1}{\pi \, \mathrm{L}(\partial\Omega)} \int_0^{\mathrm{L}(\partial\Omega)} \int_0^{\pi} \sigma_1 \, d\theta \, ds$$

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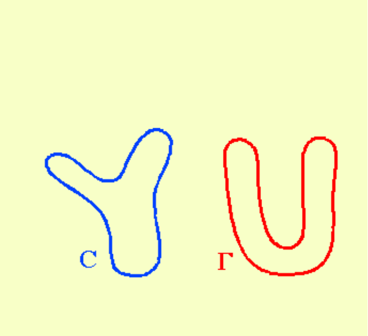
3. Two uniform random points on the boundary
$$\mathbb{E}_3(\sigma_1) = \frac{1}{\mathrm{L}(\partial\Omega)^2} \int_0^{\mathrm{L}(\partial\Omega)} \int_0^{\mathrm{L}(\partial\Omega)} \sigma_1 \, ds_1 \, ds_2$$

What is the average length of a chord of a compact convex set Ω?
There are many answers. Depends on what "random line" means.

When Ω is disk of radius R,

1. Uniform distance from origin and uniform angle (proportional to dK)
$$\mathbb{E}(\sigma_1) = \frac{\int_{\{L: L\cap\partial\Omega\neq\emptyset\}} \sigma_1 \, dK}{\int_{\{L: L\cap\partial\Omega\neq\emptyset\}} dK} = \frac{\pi \,\mathsf{A}(\Omega)}{\mathsf{L}(\partial\Omega)} = \frac{\pi R}{2}$$
2. Uniform point on boundary and uniform angle
$$\mathbb{E}_2(\sigma_1) = \frac{1}{\pi \,\mathsf{L}(\partial\Omega)} \int_0^{\mathsf{L}(\partial\Omega)} \int_0^{\pi} \sigma_1 \, d\theta \, ds = \frac{4R}{\pi}$$
3. Two uniform random points on the boundary
$$\mathbb{E}_3(\sigma_1) = \frac{1}{\mathsf{L}(\partial\Omega)^2} \int_0^{\mathsf{L}(\partial\Omega)} \int_0^{\mathsf{L}(\partial\Omega)} \sigma_1 \, ds_1 \, ds_2 = \frac{4R}{\pi}$$



Let C and Γ be two piecewise $\mathcal{C}^1$ curves in the plane. Using rigid motion, we move Γ around the plane

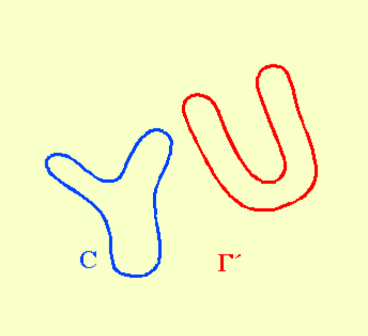
$$\Gamma' = \mathcal{M}_{a,b,\phi}(\Gamma).$$

$\mathcal{M}_{a,b,\phi}$ is rotation by angle ϕ followed by translation by vector (a, b)

$$x' = x\cos\phi - b\sin\phi + a$$
$$y' = x\sin\phi + y\cos\phi + b$$

The Kinematic Density is the invariant measure on motions of Γ' given by

$$dK = da \wedge db \wedge d\phi.$$



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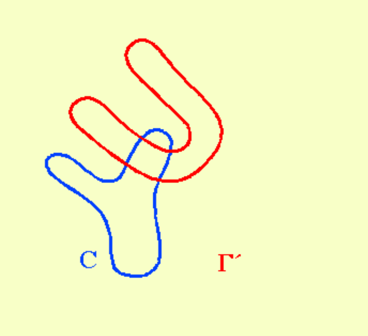

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The Kinematic Density is the invariant measure on motions of Γ' given by

$$dK = da \wedge db \wedge d\phi.$$

Theorem (Poincaré's Formula for intersecting curves [1912])

Let C and Γ be piecewise $\mathcal{C}^1$ curves in the plane. Let $n(C \cap \Gamma')$ denote the number of intersection points between C and a moving Γ'. Then

$$\int_{\{\Gamma' : C \cap \Gamma' \neq \emptyset\}} n(C \cap \Gamma')\, dK(\Gamma') = 4\, \mathrm{L}(C)\, \mathrm{L}(\Gamma).$$

We show the formula for $\mathcal{C}^1$ curves and add to get it for piecewise $\mathcal{C}^1$ curves. We give two computations of the integral over the "flag" subset

$$\mathcal{S} = \{(\Gamma', X) : C \cap \Gamma' \neq \emptyset, \quad X \in C \cap \Gamma'\}.$$

For simplicity, suppose the origin $0 \in C$ and $0 \in \Gamma$.

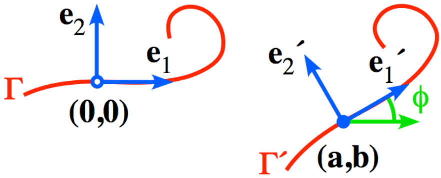

Figure: Attach a unit frame to the moving curve.

Let $\mathcal{I}$ be the integral over $\mathcal{S}$ the first way.

$$\mathcal{I} = \int_{\{\Gamma' : C \cap \Gamma' \neq \emptyset\}} n \, dK = \int_{\{\Gamma' : C \cap \Gamma' \neq \emptyset\}} \left(\sum_{Z \in C \cap \Gamma'} 1 \right) da \, db \, d\phi \qquad (7)$$

For the second equivalent way, we pick a point Z common to both curves first and then the angle ψ between the tangents of C and Γ'.

32. Finish the proof of Poincaré's Formula.

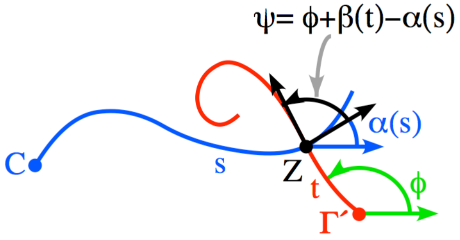

Figure: Angle between C and γ' at Z.

Let s be arclength along C from the origin and t arclength along Γ from the origin corresponding to the common point $Z \in C \cap \Gamma'$. Let $\alpha(s)$ denote the tangent angle at $(x(s), y(s)) \in C$ and $\beta(t)$ the tangent angle at $(u(t), v(t)) \in \Gamma$. The coordinates (x, y) of Z are given in two ways

$$x(s) = a + u(t)\cos\phi - v(t)\sin\phi$$
$$y(s) = b + u(t)\sin\phi + v(t)\cos\phi$$
$$\psi = \phi + \beta(t) - \alpha(s)$$

33. Finish the proof of Poincaré's Formula-.

Change to (s, t, ψ) coordinates for $\mathcal{S}$. Differentiating the defining equations,

$$\begin{aligned}
\dot{x}(s)\,ds &= da + \big[\dot{u}(t)\cos\phi - \dot{v}(t)\sin\phi\big]\,dt - \big[u(t)\sin\phi + v(t)\cos\phi\big]\,d\phi \\
\dot{y}(s)\,ds &= db + \big[\dot{u}(t)\sin\phi + \dot{v}(t)\cos\phi\big]\,dt + \big[u(t)\cos\phi - v(t)\sin\phi\big]\,d\phi \\
d\psi &= d\phi + \dot{\beta}(t)\,dt - \dot{\alpha}(s)\,ds
\end{aligned}$$

Using $(\cos\alpha, \sin\alpha) = (\dot{x}, \dot{y})$ and $(\cos\beta, \sin\beta) = (\dot{u}, \dot{v})$, the kinematic density is thus $da \wedge db \wedge d\phi$

$$\begin{aligned}
&= \Big[\dot{x}(s)\,ds - \big[\dot{u}(t)\cos\phi - \dot{v}(t)\sin\phi\big]\,dt + \big[u(t)\sin\phi + v(t)\cos\phi\big]\,d\phi\Big] \\
&\quad \wedge \Big[\dot{y}(s)\,ds - \big[\dot{u}(t)\sin\phi + \dot{v}(t)\cos\phi\big]\,dt - \big[u(t)\cos\phi - v(t)\sin\phi\big]\,d\phi\Big] \\
&\quad \wedge \Big[d\psi - \dot{\beta}(t)\,dt + \dot{\alpha}(s)\,ds\Big] \\
&= \big(-\dot{x}\big[\dot{u}\sin\phi + \dot{v}\cos\phi\big] + \dot{y}\big[\dot{u}\cos\phi - \dot{v}\sin\phi\big]\big)\,ds \wedge dt \wedge d\psi \\
&= -\sin(\psi)\,ds \wedge dt \wedge d\psi.
\end{aligned}$$

Using Fubini's theorem, we find another expression for (7)

$$\mathcal{I} = \int_C \int_\Gamma \int_0^{2\pi} da\, db\, d\phi = \int_C \int_\Gamma \int_0^{2\pi} |\sin(\psi)|\, d\psi\, dt\, ds = 4\,\mathsf{L}(C)\,\mathsf{L}(\Gamma). \qquad \square$$

Figure: Luis Santaló 1911-2001.

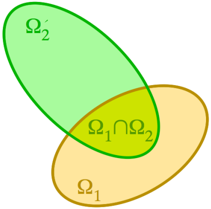

Figure: Convex domains have convex intersection.

Theorem (Santaló's Formula for convex domains [1935])

Let Ω_1 and Ω_2 be convex plane domains. We assume that Ω_2' is moving in the plane with kinematic density dK_2. Then

$$\int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} dK_2 = 2\pi \Big\{ \mathsf{A}(\Omega_1) + \mathsf{A}(\Omega_2) \Big\} + \mathsf{L}(\partial\Omega_1)\, \mathsf{L}(\partial\Omega_2). \tag{8}$$

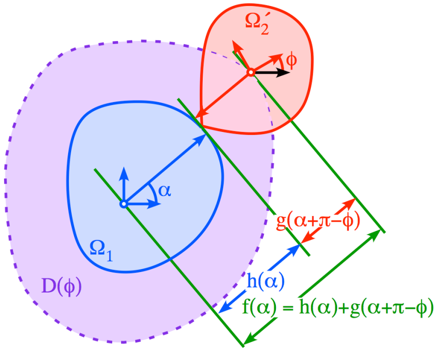

Figure: Extent D of moving center so domains overlap.

$h(\alpha)$ is the support function for Ω_1;
$g(\alpha)$ is the support function for Ω_2.

We approximate by convex sets Ω_1 and Ω_2 with piecewise $\mathcal{C}^2$ boundaries. The second domain $\Omega_2' = \mathcal{M}\Omega_2$ is moved by a rotation of angle ϕ followed by translation of vector (a, b). The kinematic density is $dK = da \wedge db \wedge d\phi$.

Fix ϕ and consider $D(\phi)$, the set of moving centers (a, b) of $\Omega_2'(\phi)$ such that the domains overlap: $\Omega_1 \cap \Omega_2'(\phi) \neq \emptyset$.

$$f(\alpha) = h(\alpha) + g(\alpha + \pi - \phi)$$

is the support function for $D(\phi)$;

Use (6) to integrate the area of the moving centers $D(\phi)$.

$$\begin{aligned}
\mathcal{J} &= \int\limits_{\{\Omega_2':\Omega_1\cap\Omega_2'\neq\emptyset\}} dK \\
&= \int\limits_0^{2\pi} \int_{\{\Omega_2'(\phi):\Omega_1\cap\Omega_2'(\phi)\neq\emptyset\}} da\, db\, d\phi \\
&= \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f(\alpha)\Big[f(\alpha)+\ddot{f}(\alpha)\Big]\, d\alpha\, d\phi \\
&= \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{2\pi} [h(\alpha)+g(\alpha+\pi-\phi)] \begin{bmatrix} h(\alpha)+g(\alpha+\pi-\phi) \\ +\ddot{h}(\alpha)+\ddot{g}(\alpha+\pi-\phi) \end{bmatrix} d\alpha\, d\phi
\end{aligned}$$

Using Fubini's theorem, Cauchy's Formula (5) and $\int_0^{2\pi} \ddot{h}(\alpha)\, d\alpha = 0$,

$$
\begin{aligned}
2\mathcal{J} = & \int_0^{2\pi} \int_0^{2\pi} h(\alpha) \left[h(\alpha) + \ddot{h}(\alpha) \right] \, d\alpha \, d\phi \\
& + \int_0^{2\pi} \int_0^{2\pi} g(\alpha + \pi - \phi) \left[g(\alpha + \pi - \phi) + \ddot{g}(\alpha + \pi - \phi) \right] \, d\alpha \, d\phi \\
& + \int_0^{2\pi} \int_0^{2\pi} h(\alpha) \left[g(\alpha + \pi - \phi) + \ddot{g}(\alpha + \pi - \phi) \right] \, d\phi \, d\alpha \\
& + \int_0^{2\pi} \int_0^{2\pi} g(\alpha + \pi - \phi) \left[h(\alpha) + \ddot{h}(\alpha) \right] \, d\phi \, d\alpha \\
= {} & 4\pi \, \mathsf{A}(\Omega_1) + 4\pi \, \mathsf{A}(\Omega_2) \\
& + \int_0^{2\pi} h(\alpha) \Big[\mathsf{L}(\partial\Omega_2) + 0 \Big] \, d\alpha + \int_0^{2\pi} \mathsf{L}(\partial\Omega_2) \left[h(\alpha) + \ddot{h}(\alpha) \right] \, d\alpha \\
= {} & 4\pi \, \mathsf{A}(\Omega_1) + 4\pi \, \mathsf{A}(\Omega_2) + \mathsf{L}(\partial\Omega_1) \, \mathsf{L}(\partial\Omega_2) + \mathsf{L}(\partial\Omega_2) \Big[\mathsf{L}(\partial\Omega_1) + 0 \Big].
\end{aligned}
$$

□

Corollary

Let Ω_1 and Ω_2 be bounded convex planar domains. The expected number of intersections of $\partial\Omega_1$ with a moving $\partial\Omega_2'$ given that Ω_2' meets Ω_1 is

$$\mathbb{E}(\mathsf{n}) = \frac{4\,\mathsf{L}(\partial\Omega_1)\,\mathsf{L}(\partial\Omega_2)}{2\pi\Big\{\mathsf{A}(\Omega_1)+\mathsf{A}(\Omega_2)\Big\}+\mathsf{L}(\partial\Omega_1)\,\mathsf{L}(\partial\Omega_2)}.$$

If $\Omega_1 \cong \Omega_2$ are congruent, then $\mathbb{E}(\mathsf{n}) \ge 2$ with "$=$" iff Ω_1 is a circle.

Proof. Apply Poincaré's and Santaló's Formulas to the expectation

$$\mathbb{E}(\mathsf{n}) = \frac{\int_{\{\partial\Omega_2' : \partial\Omega_1\cap\partial\Omega_2'\ne\emptyset\}} \mathsf{n}(\partial\Omega_2'\cap\partial\Omega_2')\,dK}{\int_{\{\Omega_2' : \Omega_1\cap\Omega_2'\ne\emptyset\}} dK_2}.$$

If $\Omega_1 \cong \Omega_2$ are congruent, the isoperimetric inequality implies
$\mathbb{E}(\mathsf{n}) = \dfrac{4\,\mathsf{L}^2}{4\pi\,\mathsf{A}+\mathsf{L}^2} \ge \dfrac{4\,\mathsf{L}^2}{\mathsf{L}^2+\mathsf{L}^2} = 2$ with equality iff Ω_1 is circle. □

Let C be closed piecewise $\mathcal{C}^2$ curve. The curvature is $\kappa = \dfrac{\partial\alpha}{\partial s}$, the rate of turning, where α gives the angle via $(\cos\alpha, \sin\alpha) = \dot{Z}$, the direction of C at Z.

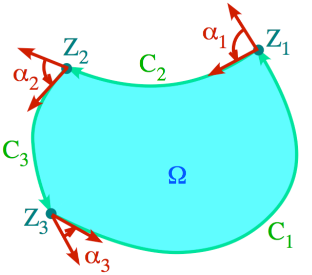

Figure: Piecewise $\mathcal{C}^2$ boundary with corners at Z_i

A piecewise $\mathcal{C}^2$ boundary is the union of n curves $\partial\Omega = \bigcup_{i=1}^{n} C_i$.

The total curvature is the integral of the curvatures over the $\mathcal{C}^2$ curves C_i plus the turning angle at the vertices Z_i between C_i and C_{i+1}

$$\mathrm{c}(\partial\Omega) = \sum_{i=1}^{n} \int_{C_i} \kappa\, ds + \sum_{i=1}^{n} \alpha_i$$

By the Gauss-Bonnet Formula, the total curvature of a boundary is related to the Euler Characteristic

$$\mathrm{c}(\partial\Omega) = 2\pi\chi(\Omega).$$

Figure: Wilhelm Blaschke 1885–1962

Theorem (Blashke's Fundamental Formula [1936])

Let Ω_1 and Ω_2 be plane domains bounded by finitely many oriented, piecewise $\mathcal{C}^2$, simple, closed curves. We assume that Ω_2' is moving in the plane with kinematic density dK_2. Then

$$\int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} \mathsf{c}(\Omega_1 \cap \Omega_2')\, dK_2 = 2\pi \left\{ \begin{matrix} \mathsf{A}(\Omega_1)\,\mathsf{c}(\Omega_2) + \mathsf{A}(\Omega_2)\,\mathsf{c}(\Omega_1) \\ + \mathsf{L}(\partial\Omega_1)\,\mathsf{L}(\partial\Omega_2) \end{matrix} \right\}.$$

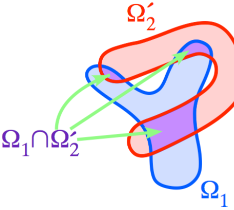

Figure: Simple boundaries: count components of intersection.

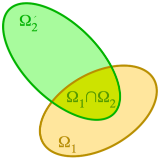

Figure: Convex domains have convex intersection.

Case 1. Both domains bounded by one simple curve. Then $c(\Omega_i) = 2\pi$. Let $\nu(\Omega_1 \cap \Omega_2')$ be number of components.

$$\int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} \nu(\Omega_1 \cap \Omega_2') \, dK_2$$
$$= 2\pi \left\{ A(\Omega_1) + A(\Omega_2) \right\} + L(\partial\Omega_1) \, L(\partial\Omega_2).$$

Case 2. Both domains convex. Then $\nu(\Omega_1 \cap \Omega_2) = 1$. We recover (8):

$$\int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} dK_2$$
$$= 2\pi \left\{ A(\Omega_1) + A(\Omega_2) \right\} + L(\partial\Omega_1) \, L(\partial\Omega_2).$$

Among all domains in the plane with a fixed boundary length, the circle has the greatest area. For simplicity we focus on domains bounded by simple curves.

Theorem (Isoperimetric Inequality.)

1. *Let C be a simple closed curve in the plane whose length is L and that encloses an area A. Then the following inequality holds*

$$4\pi A \leq L^2. \tag{9}$$

2. *If equality holds in (9), then the curve C is a circle.*

Simple means curve is assumed to have no self intersections.
A circle of radius r has $L = 2\pi r$ and encloses $A = \pi r^2 = \frac{L^2}{4\pi}$.
Thus the isoperimetric Inequality says if C is a simple closed curve of length L and encloses an area A, then C encloses an area no bigger than the area of the circle with the same length.

A set $K \subset \mathbf{E}^2$ is convex if for every pair of points $x, y \in K$, the straight line segment $\overline{xy}$ from x to y is also in K, *i.e.,* $\overline{xy} \subset K$.
The bounding curve of a convex set is automatically rectifiable. The convex hull of K, denoted $\hat{K}$, is the smallest convex set that contains K. This is equivalent to the intersection of all halfspaces that contain K,

$$\hat{K} = \bigcap_{\substack{\Omega \text{ is convex} \\ \Omega \supset K}} \Omega = \bigcap_{\substack{H \text{ is a halfspace} \\ H \supset K}} H.$$

A halfspace is a set of the form $H = \{(x, y) \in \mathbf{E}^2 : ax + by \leq c\}$, where (a, b) is a unit vector and c is any real number.

Since $K \subset \hat{K}$ by its definition, we have $\mathsf{A}(\hat{K}) \geq \mathsf{A}(K)$.
Taking convex hull reduces the boundary length because the interior segments of the boundary curve, the components of $C - \partial\hat{K}$ of C are replaced by straight line segments in $\partial\hat{K}$. Thus also $\mathsf{L}(\partial\hat{K}) \leq \mathsf{L}(\partial K)$.

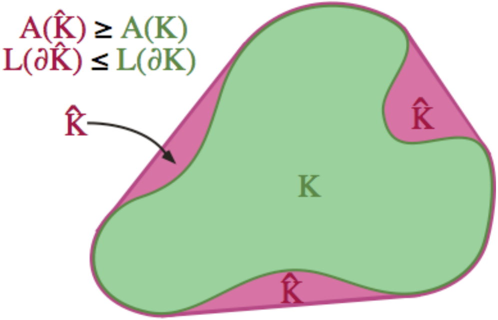

Figure: The region K and its convex hull $\hat{K}$.

Thus the isoperimetric inequality for convex sets implies

$$4\pi A \le 4\pi \hat{A} \le \hat{L}^2 \le L^2.$$

Furthermore, one may also argue that equality $4\pi A = L^2$ implies equality $4\pi \hat{A} = \hat{L}^2$ in the isoperimetric inequality for convex sets so that $\hat{K}$ is a circle. But then so is K.

The basic idea is to consider the the extreme points $\partial^* \hat{K} \subset \partial \hat{K}$ of $\hat{K}$, that is points $x \in \partial \hat{K}$ such that if $x = \lambda y + (1 - \lambda) z$ for some $y, z \in \hat{K}$ and $0 < \lambda < 1$ then $y = z = x$. $\hat{K}$ is the convex hull of its extreme points. However, the extreme points of the convex hull lie in the curve $\partial^* \hat{K} \subset C \cap \partial \hat{K}$. $\hat{K}$ being a circle implies that every boundary point is an extreme point, and since they come from C, it means that C is a circle.

There are many proofs of the isoperimetric inequality. We shall give two integral geometric arguments due to Luis Santaló.

1. The first argument only establishes the inequality part $4\pi A \le L^2$.
2. To show that the circle is the unique domain for which the Isoperimetric Inequality is equality, we prove a strong isoperimetric inequality (12) that follows from Bonnesen's inequality (11). The second argument is Santaló's proof of Bonnesen's inequality.

Lemma (Isoperimetric Inequality for convex sets.)

If Ω is a convex plane domain with boundary length L and area A, then

$$4\pi A \le L^2. \tag{10}$$

Proof. Let Ω_1 and Ω_2 be congruent copies of Ω. Let m_i denote the measure of positions of a moving Ω_2' for which the number of intersections

$$\mathrm{n}(\partial\Omega_1 \cap \partial\Omega_2') = i.$$

Note that positions that have an odd or infinite number of intersection points is dK-measure zero so that

$$m_i = 0 \quad \text{if } i \text{ is odd.}$$

49. Finish Santaló's proof of the Isoperimetric Inequality.

Then by Poincaré's and Santaló's formulas,

$$4\,\mathsf{L}(\partial\Omega)^2 = \int_{\{\Omega_2' : \partial\Omega_2' \cap \partial\Omega_1 \neq \emptyset\}} \mathsf{n}(\partial\Omega_1 \cap \partial\Omega_2')\, dK = 2m_2 + 4m_4 + 6m_6 + \cdots,$$

$$4\pi\,\mathsf{A}(\Omega) + \mathsf{L}(\partial\Omega)^2 = \int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} dK = m_2 + m_4 + m_6 + \cdots.$$

Subtracting,

$$\mathsf{L}(\partial\Omega)^2 - 4\pi\,\mathsf{A}(\Omega) = m_4 + 2m_6 + 3m_8 + \cdots \geq 0,$$

since all the measures $m_i \geq 0$. □

Let K be the region bounded by γ. The radius of the smallest circular disk containing K is called the circumradius, denoted R_{out}. The radius of the largest circular disk contained in K is the inradius.

$$R_{\text{in}} = \sup\{r : \text{there is } p \in \mathbf{E}^2 \text{ such that } B_r(p) \subseteq K\}$$

$$R_{\text{out}} = \inf\{r : \text{there exists } p \in \mathbf{E}^2 \text{ such that } K \subseteq B_r(p)\}$$

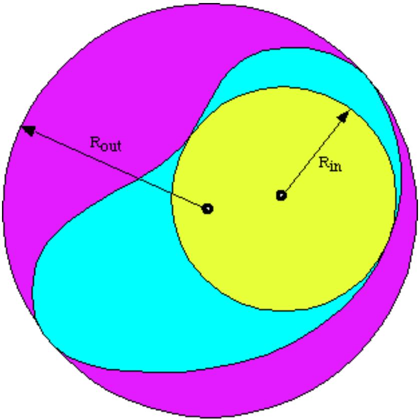

Figure: T. Bonnesen 1873–1935

Theorem (Bonnesen's Inequality [1921])

Let Ω be a convex plane domain whose boundary has length L and whose area is A. Let R_{in} and R_{out} denote the inradius and circumradius of the region Ω. Then

$$sL \geq A + \pi s^2 \text{ for all } R_{in} \leq s \leq R_{out}. \qquad (11)$$

Bonnesen's strong isoperimetric inequality follows immediately.

Corollary (Strong Isoperimetric Inequality of Bonnesen)

Let Ω be a convex planar domain with boundary length L and area A. Let R_{in} and R_{out} denote the inradius and circumradius of the Ω. Then

$$L^2 - 4\pi A \geq \pi^2 (R_{out} - R_{in})^2. \qquad (12)$$

Proof of corollary. Consider the quadratic function $f(s) = \pi s^2 - Ls + A$. By Bonnesen's inequality, $f(s) \leq 0$ for all $R_{\text{in}} \leq s \leq R_{\text{out}}$. Hence these numbers are located between the zeros of $f(s)$, namely

$$R_{\text{out}} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}$$
$$\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq R_{\text{in}}.$$

Subtracting these inequalities gives

$$R_{\text{out}} - R_{\text{in}} \leq \frac{\sqrt{L^2 - 4\pi A}}{\pi},$$

which is (12). □

53. Strong Isoperimetric Inequality implies the Isoperimetric Inequality

Obvious. The strong isoperimetric inequality (12) implies part one of the isoperimetric inequality (10), since $\pi^2(R_{\text{out}} - R_{\text{in}})^2 \geq 0$.

Moreover, if equality holds in (9), then $L^2 - 4\pi A = 0$ which implies that $R_{\text{in}} = R_{\text{out}}$, or Ω is a circle. □

Theorem (Bonnesen's Inequality)

Let Ω be a bounded convex plane domain whose boundary has length L and whose area is A. Let R_{in} and R_{out} be the inradius and circumradius of the region Ω. Then $sL \geq A + \pi s^2$ for all $R_{in} \leq s \leq R_{out}$.

Proof. Let $\Omega_1 = \Omega$ and Ω_2' be a moving circular disk of radius s. Because $R_{\text{in}} \leq s \leq R_{\text{out}}$, the sets overlap, $\Omega_1 \cap \Omega_2' \neq \emptyset$, if and only if their boundaries overlap, $\partial\Omega_1 \cap \partial\Omega_2' \neq \emptyset$, hence the Poincaré and Blaschke integrals are taken over the same positions of Ω_2'.

As before, let m_i denote the measure of positions of the moving Ω_2' for which the number of intersections $\text{n}(\partial\Omega_1 \cap \partial\Omega_2') = i$, *i.e.*,

$$m_i = dK\left(\left\{\Omega_2' : \text{n}(\partial\Omega_1 \cap \partial\Omega_2') = i\right\}\right).$$

Again, positions that have an odd or infinite number of intersection points is dK-measure zero so that $m_i = 0$ if i is odd.

Then by Poincaré's and Santaló's formulas,

$$8\pi s\,\mathsf{L}(\partial\Omega) = \int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} \mathsf{n}(\partial\Omega_1 \cap \partial\Omega_2')\, dK = 2m_2 + 4m_4 + 6m_6 + \cdots,$$

$$2\pi\,\mathsf{A}(\Omega) + 2\pi^2 s^2 + 2\pi s\,\mathsf{L}(\partial\Omega) = \int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} dK = m_2 + m_4 + m_6 + \cdots.$$

Subtracting,

$$2\pi\Big(s\,\mathsf{L}(\partial\Omega) - \mathsf{A}(\Omega) - \pi s^2\Big) = m_4 + 2m_6 + 3m_8 + \cdots \geq 0,$$

since all the measures $m_i \geq 0$. □

Thanks!